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Integrability of the field equations of invariant variational problems on linear frame bundles

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Abstract

The integrability of the Euler–Lagrange equations and the Jacobi fields of the natural basis of Lagrangian densities on the bundle of linear frames of a manifold which are invariant under diffeomorphisms, is stated. Applications to the reducibility of the pre-symplectic structure attached to such variational problems as defined in [Symp. Math. 14 (1974) 219], are also given.

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1. Introduction

The main aim of this paper is to prove the integrability of the field equations and their Jacobi fields for the variational problems defined by the natural basis of diffeomorphism-invariant Lagrangians on the linear frame bundle of a manifold. To this end, we use the tools of geometric theory of partial differential equations. Formal integrability, existence of quasi-regular bases (and hence, the vanishing of the Spencer cohomology groups; e.g.,

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see [2, IX, Theorem 2.14]), etc. The application of these techniques to the equations of the Mathematical Physics stars with Einstein's equations (e.g., see [6]) and more recently in studying Yang–Mills fields; see [8] and also [4,7] for general expositions on the topic.

Let $\pi : FM \to M$ be the bundle of linear frames. Every Diff M-invariant Lagrangian on $J^1(FM)$ can be written as a differentiable function of the Lagrangians $\mathcal{L}^i_{jk}$ defined by $(\mathcal{L}^i_{jk} \circ j^1 s) X_i = [X_j, X_k]$, where $s = (X_1, \dots, X_m)$ is a linear frame (Latin indices run from 1 to $m = \dim M$); i.e., $\mathcal{L}^i_{jk}$ associates the ith component of $[X_j, X_k]$ in the given frame to the 1-jet prolongation of s (see [16]). Denoting by $\theta = (\theta^1, \dots, \theta^m)$ the canonical form on FM (see Section 2.1), the Lagrangian densities $\Omega^i_{jk} = \mathcal{L}^i_{jk} \theta^1 \wedge \cdots \wedge \theta^m$ define the simplest class of diffeomorphism-invariant variational problems; they do not apply immediately to field theory but they provide interesting geometric relativistic models. The bundle of linear frames seems to be the natural framework for formulating Diff M-invariant problems in the affine gauge theory of gravity; e.g., see [10] and also [18,19] for a discussion of the central role of that bundle in classical field theory. The present work was originally motivated by the analysis of the consistency of models of relativistic Lagrangians on the linear frame bundle. In fact, the formal integrability of Euler–Lagrange equations solves, in particular, the Cauchy problem, and this is a good measure of the existence of solutions. In [16] we have studied the Hamiltonian structure associated to the variational problems defined by the densities Ω^i_{jk}. The results in this paper allow us to determine the reduction of the pre-symplectic structure attached to such problems in the sense of García [5], defined by means of the exterior differential of the Poincaré–Cartan as an alternating bilinear map on the space of Jacobi fields along a given extremal. This represents a novel application of the geometric techniques to study the symplectic structure on the space of extremals.

The outline of the paper is as follows. In Section 2 we briefly introduce the geometric notions on linear frame bundles, jet bundles and quasi-linear PDEs that we use throughout the paper, as well as the basis of invariant Lagrangians on FM and the equations defining the extremals of Ω^1_{23} and Ω^1_{12}. For more details we refer the reader to [16]. In Section 3 we state the formal integrability of the extremals of Ω^1_{23} and Ω^1_{12} (Theorem 3.13) as well as that of Jacobi fields along an extremal (Theorem 3.25). We remark that, the field equations being non-linear, the integrability of Jacobi fields is not a consequence of the integrability of Euler–Lagrange equations. As the equations defining the extremals of Ω^1_{23} and Ω^1_{12} are locally of class C^ω, their formal integrability implies local integrability (Corollary 3.14), and similarly for Jacobi fields (Theorem 3.25). Two features of the field equations (1) and (2) studied below are the following. They are underdetermined (cf. [1]) quasi-linear systems of PDEs and every hypersurface of M is characteristic for such equations (Proposition 3.6). This explains why these systems cannot be written in the Cauchy–Kovalevskaya form; these are the contents of Sections 3.1 and 3.2. In Section 3.3 we prove that the curvature of the Euler–Lagrange equations vanishes (Theorems 3.8 and 3.9) and in Section 3.4, the existence of quasi-regular bases is stated (Theorems 3.11 and 3.12). In Section 3.5, we prove the formal integrability of the Jacobi equations along an extremal of class C^ω, which implies its local integrability (Theorem 3.25). In Section 4 we assume that M is parallelizable and that Ω^1_{23} and Ω^1_{12} admit global extremals. In Section 4.1, by using the integrability of the Jacobi fields, we characterize the radical of the pre-symplectic structure attached to Ω^1_{23} (resp. Ω^1_{12}) as the set of the Jacobi fields along an extremal which are infinitesimal symmetries of Ω^1_{23} (resp. Ω^1_{12}) (Theorem 4.1). Here we consider that the pre-symplectic structure takes values into the space of closed $(m-1)$-forms on the ground manifold M. On the other hand, in Section 4.2, we analyse the radical of the pre-symplectic structures of Ω^1_{23} and Ω^1_{12} modulo exact $(m-1)$-forms; i.e., we consider the pre-symplectic structure as an $H^{m-1}(M;\mathbb{R})$-valued 2-form. First of all, in Section 4.2.1, we discuss the case dim $M = 3$ and we conclude (Theorem 4.3) that the only obstruction for the pre-symplectic structure to vanish, lies in $H^1(M;\mathbb{R})$. We remark that no compactness assumption on M is supposed. Finally, in Section 4.2.2, we study the radical of the pre-symplectic structure at a global holonomic section for dim $M \geq 4$; hence we need to assume (cf. [12]) that our domain of integration is an $(m-1)$-dimensional torus.

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2. Notations and preliminaries

2.1. The bundle of linear frames

Let $\pi : FM \to M$ be the bundle of linear frames of a connected m-dimensional manifold M. Let $T = T(M)$ and $T^* = T^*(M)$ be the tangent bundle and cotangent bundle of M, respectively. Each coordinate system $(x^1, \dots, x^m)$ on an open domain $U \subseteq M$ induces a coordinate system (x^i, x^i_j), $i, j = 1, \dots, m$, on $\pi^{-1}(U)$ by setting $u = ((\partial/\partial x^1)_x, \dots, (\partial/\partial x^m)_x)\cdot(x^i_j(u))$, $x = \pi(u)$. As is well known (e.g., see [13, VI, Section 1]) every local diffeomorphism $\phi : M \to M'$ induces a principal bundle morphism $\tilde{\phi} : FM \to FM'$, $\tilde{\phi}(X_1, \dots, X_m) = (\phi_* X_1, \dots, \phi_* X_m)$. If ϕ_t is the local flow of $Z \in \mathfrak{X}(M)$, then $\tilde{\phi}_t$ is the local flow of a vector field $\tilde{Z} \in \mathfrak{X}(FM)$ (see [13, VI, Proposition 2.1]).

We denote by $\theta = (\theta^1, \dots, \theta^m)$, $\theta^i = x_j{}^i\, \mathrm{d}x^j$, $1 \leq i \leq m$, the canonical 1-form on FM (see [13, III, Section 2]), where $(x_j{}^i) = (x^i_j)^{-1}$ is the inverse matrix.

2.2. Jet bundle notations

Let $\pi : P \to M$ be a fibred manifold; i.e., π is a surjective submersion. We denote by $\pi^r : J^r P \to M$ the bundle of r-jets of local sections of π, and by $j^r s : U \to J^r P$ the r-jet prolongation of a section $s : U \to P$ of π on an open subset $U \subseteq M$. For $r \geq k$ there is a projection $\pi^r_k : J^r P \to J^k P$ given by $\pi^r_k(j^r_x s) = j^k_x s$. Set dim $P = m + n$. Throughout the paper Latin indices run from 1 to m, and Greek indices run from 1 to n. A fibred coordinate system for π is a coordinate system $(V; x^i, y^\alpha)$ on an open subset $V \subseteq P$ such that (x^i) is a coordinate system on $\pi(V) = U \subseteq M$. We denote by $(x^i, y^\alpha, y^\alpha_I)$, the induced coordinate system on $J^r V$; i.e., $y^\alpha_I(j^r_x s) = (\partial^{|I|}(y^\alpha \circ s)/\partial x^I)(x)$, where $I = (i_1, \dots, i_m)$ is a multi-index of order $|I| = i_1 + \cdots + i_m \leq r$. We denote by (i) the multi-index whose entries are defined by $(i)_h = \delta^i_h$, for $1 \leq h \leq m$. We also set $(ij) = (i) + (j)$, $(ijk) = (i) + (j) + (k)$, etc. Remark that the symbols (ij), (ijk), etc. depend symmetrically on the indices i, j, k.

For linear frame bundles $\pi : FM \to M$ we use a specific notation: We denote by $(x^i, x^i_j; x^i_{j,I})$ the coordinate system on $J^r(FM)$ induced by a coordinate system (x^i) in M; i.e., $x^i_{j,I}(j^1_x s) = (\partial^{|I|}(x^i_j \circ s)/\partial x^I)(x)$.

If $\pi : E \to M$ is a vector bundle then $\pi^r : J^r E \to M$ is a vector bundle with respect to the operations $j^r_x s + j^r_x s' = j^r_x(s + s')$, $\lambda \cdot j^r_x s = j^r_x(\lambda s)\, \forall j^r_x s,\ j^r_x s' \in J^r_x E$, $\lambda \in \mathbb{R}$ (see [9] for details). There exists an exact sequence of vector bundles over M (cf. [2, p. 395], [9, p. 6]), $0 \to S^r T^* \otimes E \xrightarrow{\varepsilon} J^r E \to J^{r-1}E \to 0$, where the injection ε is defined as follows: If $[f]^r_x$ denotes the coset of $f \in \mathfrak{m}^r_x$ in $\mathfrak{m}^r_x/\mathfrak{m}^{r+1}_x \cong S^r T^*_x$, $\mathfrak{m}_x = \{f \in C^\infty(M) | f(x) = 0\}$, and $v = (v_1, \dots, v_m) \in E_x$, then $\varepsilon([f]^r_x \otimes v) = j^r_x(fY_1, \dots, fY_m)$, where Y_i is any vector field such that $(Y_i)_x = v_i$. Hence, the vector bundle $S^r T^* \otimes E$ can be identified to the sub-bundle in $J^r E$ of the elements $j^r_x s'$ such that $(\partial^{|I|} s'/\partial x^I)(x) = 0$ for $|I| \leq r - 1$.

In the case $E = \oplus^m T$, let denote $(x^i, x^i_j; x^i_{j,I})$ the coordinate system on $J^r E$ defined similarly as in $J^r(FM)$. The injection $\varepsilon : S^r T^* \otimes E \to J^r E$ is given by $\varepsilon^* x^i_j = 0$, $\varepsilon^* x^i_{j,I} = 0$ for $|I| \leq r - 1$ and $\varepsilon^* x^i_{j,I} = t^i_{j,I}$ for $|I| = r$, where $t^i_{j,I}$ are the induced coordinates on $S^r T^* \otimes E$; i.e., $t^i_{j,I}([f]^r_x \otimes v) = (\partial^r f/\partial x^I)(x) x^i_j(v)$.

2.3. Quasi-linear PDEs

Let $\pi : P \to M$ be a fibred manifold. A partial differential equation of order k (see [2]) is a fibred submanifold $R^k \subset J^k P$; a solution of R^k is a section $s : U \to P$ such that $j^k_x s \in R^k\, \forall x \in U$. If $\varphi : J^k P \to P'$ is a morphism of fibred manifolds of locally constant rank and $s' : M \to P'$ is a global section of π', then $R^k = \{j^k_x s \in J^k P | \varphi(j^k_x s) = s'(x)\}$ is a fibred submanifold of $J^k P$.

Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles and let $\pi^k : F \to M$ be an open fibred submanifold of $J^k E$. In our case $F = J^1(FM)$, considered as an open fibred submanifold in $J^1(\oplus^m T)$. A morphism $\varphi : F \to E'$ is said to be quasi-linear if a morphism $\sigma = \sigma(\varphi) : \pi^k_{k-1}(F) \times_M (S^k T^* \otimes E) \to E'$ of vector bundles exists—called the symbol of φ—such that for every $v \in S^k T^*_x \otimes E_x$, $j^k_x s \in F$, with $v + j^k_x s \in F$, we have $\varphi(v + j^k_x s) = \sigma(j^{k-1}_x s, v) + \varphi(j^k_x s)$. Let $s' : M \to E'$ be a section such that $s'(M) \subset \varphi(F)$ and assume the rank of φ is constant. The PDE system $R^k = \{j^k_x s \in F | \varphi(j^k_x s) = s'(x)\}$ is then said to be quasi-linear. Moreover, there exists a quasi-linear morphism $p_l(\varphi) : (\pi^{k+l}_k)^{-1}(F) \to J^l E'$, $p_l(\varphi)(j^{k+l}_x s) = j^l_x(\varphi \circ j^k s)$, called the lth prolongation of φ, whose symbol $\sigma(p_l(\varphi)) : (\pi^{k+l-1}_k)^{-1}(F) \times_M (S^{k+l} T^* \otimes E) \to J^l E'$ is determined by the lth prolongation of the symbol of φ; for details, we refer the reader to [2, IX, Proposition 2.6].

As $\sigma(p_l(\varphi))(j^{k+l-1}_x s, v)$ belongs to $S^l T^* \otimes E'\, \forall j^{k+l-1}_x s \in (\pi^{k+l-1}_k)^{-1}(F)$, a unique morphism $\sigma_l = \sigma_l(\varphi) : (\pi^{k+l-1}_k)^{-1}(F) \times_M (S^{k+l} T^* \otimes E) \to S^l T^* \otimes E'$ exists—called the lth prolongation of σ—such that $\sigma(p_l(\varphi)) = \varepsilon' \circ \sigma_l$, where $\varepsilon' : S^l T^* \otimes E' \to J^l E'$ is the natural injection (cf. Section 2.2).

For every integer $l \geq 1$, we denote by $R^{k+l} \subseteq J^{k+l} P$ the lth prolongation of R^k. If $\Omega : R^k \to (T^* \otimes E')/\mathrm{im}\, \sigma_1$, $\Omega(j^k_x s) = (p_1(\varphi)(j^{k+1}_x s) - j^1_x s') \bmod \mathrm{im}\, \sigma_1$ denotes the curvature of R^k (e.g., see [2, IX, Proposition 2.8]), then we have $\pi^{k+1}_k(R^{k+1}) = \{j^k_x s \in R^k | \Omega(j^k_x s) = 0\}$. Hence $\pi^{k+1}_k : R^{k+1} \to R^k$ is surjective if and only if the curvature of R^k vanishes.

We set $\mathfrak{g}_k = \ker\sigma$, $\mathfrak{g}_{k+l} = \ker(\sigma_l)\,\forall l \geq 0$, and we denote by $\mathfrak{g}_{k,u}$ the fibre of $\mathfrak{g}_k$ over the point $u \in FM$. Let $(t_1, \dots, t_m)$ be a basis of T_x, with dual basis $(\alpha^1, \dots, \alpha^m)$ and let $S^k T^*_{x,(t_1,\dots,t_j)}$ be the subspace of $S^k T^*_x$ generated by the symmetric products $\alpha^{i_1} \odot \cdots \odot \alpha^{i_k}$, with $j+1 \leq i_1 \leq \cdots \leq i_k \leq m$. We also set $\mathfrak{g}_{k,u,(t_1,\dots,t_j)} = \mathfrak{g}_{k,u} \cap (S^k T^*_{x,(t_1,\dots,t_j)} \otimes E_x)$. Then, $(t_1, \dots, t_m)$ is said to be a quasi-regular basis for $\mathfrak{g}_k$ at a point $u = s(x)$ (cf. [2, IX, Section 2] or [7,9]) if we have $\dim \mathfrak{g}_{k+1,u} = \dim \mathfrak{g}_{k,u} + \sum_{j=1}^{m-1} \dim \mathfrak{g}_{k,u,(t_1,\dots,t_j)}$.

2.4. Invariant Lagrangians on FM

A Lagrangian density Ω_m on $J^1(FM)$ is said to be Diff M-invariant if for every $\phi \in$ Diff M, one has $J^1(\tilde{\phi})^*\Omega_m = \Omega_m$. Let $\tilde{X}$ be the natural lift of a vector field $X \in \mathfrak{X}(M)$ to FM ([13, VI, Proposition 2.1]). A Lagrangian density Ω_m on $J^1(FM)$ is said to be $\mathfrak{X}(M)$-invariant if $L_{\tilde{X}^{(1)}}\Omega_m = 0\,\forall X \in \mathfrak{X}(M)$, where $\tilde{X}^{(1)}$ is the natural lift of $\tilde{X}$ to $J^1(FM)$ by infinitesimal contact transformations. Every Lagrangian density on $J^1(FM)$ can be written as $\Omega_m = \mathcal{L}\theta^1 \wedge \cdots \wedge \theta^m$, where θ^i are the components of the canonical 1-form and $\mathcal{L} \in C^\infty(J^1FM)$. Then, Ω_m is Diff M-invariant (resp. $\mathfrak{X}(M)$-invariant) if and only if $\mathcal{L} \circ J^1(\tilde{\phi}) = \mathcal{L}\,\forall\phi \in$ Diff M (resp. $\tilde{X}^{(1)}\mathcal{L} = 0\,\forall X \in \mathfrak{X}(M)$). This is a consequence of being θ both Diff M-invariant and $\mathfrak{X}(M)$-invariant. Hence the problem of determining invariant Lagrangian densities is reduced to that of determining invariant Lagrangian functions.

We denote by $\mathcal{I}_{\mathrm{Diff}\,M}$ (resp. $\mathcal{I}_{\mathfrak{X}(M)}$) the algebra of Diff M-invariant (resp. $\mathfrak{X}(M)$-invariant) Lagrangian functions on $J^1(FM)$. Obviously $\mathcal{I}_{\mathrm{Diff}\,M} \subseteq \mathcal{I}_{\mathfrak{X}(M)}$, and $\mathcal{I}_{\mathrm{Diff}\,M} = \mathcal{I}_{\mathfrak{X}(M)}$ except when M is orientable and admits an orientation-reversing diffeomorphism, in which case we have $\mathcal{I}_{\mathfrak{X}(M)} = \mathcal{I}_{\mathrm{Diff}\,M} \times \mathcal{I}_{\mathrm{Diff}\,M}$, $\mathcal{I}_{\mathrm{Diff}\,M}$ being identified to the diagonal of $\mathcal{I}_{\mathfrak{X}(M)}$; see [16] for details.

Let $\mathcal{L}^i_{jk} : J^1(FM) \to \mathbb{R}$, $j < k$, be the Lagrangian $\mathcal{L}^i_{jk}(j^1_x s) = \omega^i([X_j, X_k])(x)$, where $s = (X_1, \dots, X_m)$ and $(\omega^1, \dots, \omega^m)$ denotes the dual coframe. We remark that the definition makes sense as the value $\omega^i([X_j, X_k])(x)$ only depends on $j^1_x s$. From the very definition we have $[X_j, X_k]_x = \mathcal{L}^i_{jk}(j^1_x s)(X_i)_x$. The local expression of this Lagrangian in an induced coordinate system on $J^1(FM)$ is $\mathcal{L}^i_{jk} = (x^h_j x^l_{k,h} - x^h_k x^l_{j,h}) x_l{}^i$. The Lagrangians $\mathcal{L}^i_{jk}$ are seen to be Diff M-invariant and functionally independent. Hence every $\mathcal{L} \in \mathcal{I}_{\mathfrak{X}(M)}$ can be written locally as a differentiable function of these Lagrangians (see [16] for the proofs).

2.5. Extremals of $\mathcal{L}^i_{jk}$

The densities $\Omega^i_{jk} = \mathcal{L}^i_{jk}\theta^1 \wedge \cdots \wedge \theta^m$ define two types of non-equivalent variational problems according to $i \notin \{j, k\}$ or $i \in \{j, k\}$. If $\dim M = 2$, the density $\Omega^1_{12} = \mathcal{L}^1_{12}\theta^1 \wedge \theta^2$ is variationally trivial. Hence we assume $\dim M \geq 3$ and we confine ourselves to work with $\mathcal{L}^1_{23}$ and $\mathcal{L}^1_{12}$ in order to avoid unnecessary indices. In [16] we have proved that a section $s = (X_1, \dots, X_m)$ of π, $X_j = f^i_j \partial/\partial x^i$, $f^i_j \in C^\infty(U)$, defined on a coordinate domain $(U; x^i)$, is an extremal of Ω^1_{23} if and only if the following $3(m-2)$ equations hold:

$$\begin{aligned}
&\left(f_j^k \frac{\partial f_i^h}{\partial x^k} - f_i^k \frac{\partial f_j^h}{\partial x^k}\right) f_h{}^1 = 0, \quad 4 \le i \le m, \quad j = 2, 3, \\
&2\left(f_1^k \frac{\partial f_j^h}{\partial x^k} - f_j^k \frac{\partial f_1^h}{\partial x^k}\right) f_h{}^1 + \sum_{l=4}^{m} \left(f_l^k \frac{\partial f_j^h}{\partial x^k} - f_j^k \frac{\partial f_l^h}{\partial x^k}\right) f_h{}^l = 0, \quad j = 2, 3, \\
&\left(f_2^k \frac{\partial f_3^h}{\partial x^k} - f_3^k \frac{\partial f_2^h}{\partial x^k}\right) f_h{}^j = 0, \quad j \ne 2, 3.
\end{aligned} \tag{1}$$

Similarly, a section $s = (X_1, \ldots, X_m)$ of FM is an extremal of Ω^1_{12} if and only if the following $3(m-1)$ equations hold:

$$\begin{aligned}
&\left(f_j^k \frac{\partial f_i^h}{\partial x^k} - f_i^k \frac{\partial f_j^h}{\partial x^k}\right) f_h{}^1 = 0, \quad 3 \le i \le m, \quad j = 1, 2, \\
&\sum_{l=3}^{m} \left(f_l^k \frac{\partial f_j^h}{\partial x^k} - f_j^k \frac{\partial f_l^h}{\partial x^k}\right) f_h{}^l = 0, \quad j = 1, 2, \\
&\left(f_2^k \frac{\partial f_1^h}{\partial x^k} - f_1^k \frac{\partial f_2^h}{\partial x^k}\right) f_h{}^j = 0, \quad j \ne 2.
\end{aligned} \tag{2}$$

The number of the Euler–Lagrange equations for the extremals of a Lagrangian on FM is m^2, a number much greater than that of Eqs. (1) and (2) defining the extremals of Ω^1_{23}, Ω^1_{12}, respectively, but each of these systems is equivalent to the corresponding Euler–Lagrange equations, as it is proved in [16, Section 2.4].

3. Integrability of extremals and Jacobi fields

3.1. Quasi-linear PDEs defined by Ω^1_{23} and Ω^1_{12}

Proposition 3.1. *We set $E = \oplus^m T$, $E' = M \times \mathbb{R}^{3(m-2)}$. The fibred morphism $\varphi : J^1(FM) \to E'$ defined below is a submersion*:

$$\begin{aligned}
&\varphi^{i-3}(j_x^1 s) = \omega^1([X_3, X_i]_x), \qquad \varphi^{m-6+i}(j_x^1 s) = \omega^1([X_2, X_i]_x), \\
&\varphi^{2m-5}(j_x^1 s) = 2\omega^1([X_1, X_3]_x) + \sum_{h=4}^{m} \omega^h([X_h, X_3]_x), \\
&\varphi^{2m-4}(j_x^1 s) = 2\omega^1([X_1, X_2]_x) + \sum_{h=4}^{m} \omega^h([X_h, X_2]_x), \\
&\varphi^{2m-3}(j_x^1 s) = \omega^1([X_2, X_3]_x), \qquad \varphi^{2m-6+i}(j_x^1 s) = \omega^i([X_2, X_3]_x),
\end{aligned} \tag{3}$$

where $4 \le i \le m$ *and* $s = (X_1, \dots, X_m)$ *is a section of FM with dual coframe* $(\omega^1, \dots, \omega^m)$. *Hence the system* (1), *which is given by* $R^1 = \varphi^{-1}(M \times \{0\})$, *is a fibred submanifold of* $J^1(FM)$. *Furthermore,* (1) *is quasi-linear and its symbol* $\sigma : FM \times_M (T^* \otimes E) \to E'$ *is given by*

$$\begin{aligned}
\sigma^{i-3}(u, w \otimes v) &= (w \otimes v_i)((X_3)_x, (\omega^1)_x) - (w \otimes v_3)((X_i)_x, (\omega^1)_x),\\
\sigma^{m-6+i}(u, w \otimes v) &= (w \otimes v_i)((X_2)_x, (\omega^1)_x) - (w \otimes v_2)((X_i)_x, (\omega^1)_x),\\
\sigma^{2m-5}(u, w \otimes v) &= 2[(w \otimes v_3)((X_1)_x, (\omega^1)_x) - (w \otimes v_1)((X_3)_x, (\omega^1)_x)]\\
&\quad + \sum_{l=4}^{m}[(w \otimes v_3)((X_l)_x, (\omega^l)_x) - (w \otimes v_l)((X_3)_x, (\omega^l)_x)],\\
\sigma^{2m-4}(u, w \otimes v) &= 2[(w \otimes v_2)((X_1)_x, (\omega^1)_x) - (w \otimes v_1)((X_2)_x, (\omega^1)_x)]\\
&\quad + \sum_{l=4}^{m}[(w \otimes v_2)((X_l)_x, (\omega^l)_x) - (w \otimes v_l)((X_2)_x, (\omega^l)_x)],\\
\sigma^{2m-3}(u, w \otimes v) &= (w \otimes v_3)((X_2)_x, (\omega^1)_x) - (w \otimes v_2)((X_3)_x, (\omega^1)_x),\\
\sigma^{2m-6+i}(u, w \otimes v) &= (w \otimes v_3)((X_2)_x, (\omega^i)_x) - (w \otimes v_2)((X_3)_x, (\omega^i)_x), \qquad (4)
\end{aligned}$$

where $4 \le i \le m$, $u = s(x) \in F_xM$, $w \in T_x^*$ *and* $v = (v_1, \dots, v_m) \in E_x$.

Proof. The coordinate expression of φ coincides with the left-hand side of Eq. (1). Deriving with respect to $x^a_{b,c}$ yields a maximal rank matrix, hence φ is a submersion.

By using the notations introduced in Section 2.2, writing $x^i_j(w \otimes v + j^1_x s) = x^i_j(j^1_x s)$, $x^i_{j,k}(w \otimes v + j^1_x s) = t^i_{j,k}(w \otimes v) + x^i_{j,k}(j^1_x s)$, and setting $u = s(x)$, we have

$$\begin{aligned}
&\varphi^{i-3}(w \otimes v + j^1_x s)\\
&\quad = x_h{}^1(u)[(x^k_3(u)t^h_{i,k}(w \otimes v) + x^h_{i,k}(j^1_x s)) - x^k_i(u)(t^h_{3,k}(w \otimes v) + x^h_{3,k}(j^1_x s))]\\
&\quad = x_h{}^1(u)(x^k_3(u)t^h_{i,k}(w \otimes v) - x^k_i(u)t^h_{3,k}(w \otimes v)) + \varphi^{i-3}(j^1_x s)
\end{aligned}$$

for $4 \le i \le m$. Hence, $\sigma^{i-3} = \sigma^* y^{i-3} = x_h{}^1(x^k_3 t^h_{i,k} - x^k_i t^h_{3,k})$. Then $w \otimes v_j = t^i_{j,k}(w \otimes v)(\mathrm{d}x^k)_x \otimes (\partial/\partial x^i)_x$. As $(\partial/\partial x^i)_x = x_i{}^h(u)(X_h)_x$, $(\mathrm{d}x^k)_x = x^k_l(u)(\omega^l)_x$, we have $(w \otimes v_j)((X_l)_x, (\omega^h)_x) = x_i{}^h(u)x^k_l(u)t^i_{j,k}(w \otimes v)$ and $\sigma^{i-3}(u, w \otimes v) = (w \otimes v_i)((X_3)_x, (\omega^1)_x) - (w \otimes v_3)((X_i)_x, (\omega^1)_x)$. In the same way, we obtain the rest of components. □

Similarly, we have the following proposition.

Proposition 3.2. *A quasi-linear submersion* $\bar{\varphi} : J^1(FM) \to M \times \mathbb{R}^{3(m-1)}$ *of fibred manifolds on M exists such that the fibred submanifold* $\bar{R}^1 = \bar{\varphi}^{-1}(M \times \{0\})$ *defines the system* (2).

Proposition 3.3. *The first prolongation of the symbol of* $\varphi : J^1(FM) \to E'$ *on* Proposition 3.1, $\sigma_1 : J^1(FM) \times_M (S^2T^* \otimes E) \to T^* \otimes \mathbb{R}^{3(m-2)} \cong \oplus^{3(m-2)} T^*$, *is given by*

$$\begin{aligned}
(\sigma_1)^{i-3}(j^1_x s, [f]^2_x \otimes v) &= (\omega^1(Y_i)\,\mathrm{d}(X_3 f) - \omega^1(Y_3)\,\mathrm{d}(X_i f))(x),\\
(\sigma_1)^{m-6+i}(j^1_x s, [f]^2_x \otimes v) &= (\omega^1(Y_i)\,\mathrm{d}(X_2 f) - \omega^1(Y_2)\,\mathrm{d}(X_i f))(x),\\
(\sigma_1)^{2m-5}(j^1_x s, [f]^2_x \otimes v) &= 2[\omega^1(Y_3)\,\mathrm{d}(X_1 f) - \omega^1(Y_1)\,\mathrm{d}(X_3 f)](x)\\
&\quad + \sum_{l=4}^{m}[\omega^l(Y_3)\,\mathrm{d}(X_l f) - \omega^l(Y_l)\,\mathrm{d}(X_3 f)](x),\\
(\sigma_1)^{2m-4}(j^1_x s, [f]^2_x \otimes v) &= 2[\omega^1(Y_2)\,\mathrm{d}(X_1 f) - \omega^1(Y_1)\,\mathrm{d}(X_2 f)](x)\\
&\quad + \sum_{l=4}^{m}[\omega^l(Y_2)\,\mathrm{d}(X_l f) - \omega^l(Y_l)\,\mathrm{d}(X_2 f)](x),\\
(\sigma_1)^{2m-3}(j^1_x s, [f]^2_x \otimes v) &= (\omega^1(Y_3)\,\mathrm{d}(X_2 f) - \omega^1(Y_2)\,\mathrm{d}(X_3 f))(x),\\
(\sigma_1)^{2m-6+i}(j^1_x s, [f]^2_x \otimes v) &= (\omega^i(Y_3)\,\mathrm{d}(X_2 f) - \omega^i(Y_2)\,\mathrm{d}(X_3 f))(x)
\end{aligned} \tag{5}$$

for $4 \le i \le m$, *where* $s = (X_1, \dots, X_m)$ *with dual coframe* $(\omega^1, \dots, \omega^m)$, *we set* $v = ((Y_1)_x, \dots, (Y_m)_x)$, *and we have used the same notations as in* Section 2.2.

Proof. Let $p : E = \oplus^m T \to M$, $p' : E' = M \times \mathbb{R}^{3(m-2)} \to M$ be the natural projections. By using the natural identification $J^1E' \cong E' \oplus (\oplus^{3(m-2)} T^*)$, $j^1_x s' \cong (x, s'(x); (\mathrm{d}s')_x)$, the first prolongation of φ is given by $p_1(\varphi)(j^2_x s) = j^1_x(\varphi \circ j^1 s) = (\varphi(j^1_x s); \mathrm{d}(\varphi^j \circ j^1 s)(x))$, $1 \le j \le 3(m-2)$.

Let $f \in \mathfrak{m}_x$, if $X_j = f^i_j \partial/\partial x^i$, $Y_j = g^i_j \partial/\partial x^i$, one has $X_j + fY_j = h^i_j \partial/\partial x^i$, with $h^i_j = f^i_j + fg^i_j$, with $\eta^i = h_j{}^i\,\mathrm{d}x^j$ the components of the dual coframe of $\tilde{s} = (X_1 + fY_1, \dots, X_m + fY_m)$. Then $j^2_x \tilde{s} = [f]^2_x \otimes v + j^2_x s$, and we have

$$\begin{aligned}
&\sigma_1(j^1_x s, [f]^2_x \otimes v)\\
&\quad = \sigma(p_1(\varphi))(j^1_x s, [f]^2_x \otimes v) = p_1(\varphi)([f]^2_x \otimes v + j^2_x s) - p_1(\varphi)(j^2_x s)\\
&\quad = j^1_x(\varphi \circ j^1 \tilde{s}) - j^1_x(\varphi \circ j^1 s) = (\varphi(j^1_x \tilde{s}); \mathrm{d}(\varphi^j \circ j^1 \tilde{s})(x)) - (\varphi(j^1_x s); \mathrm{d}(\varphi^j \circ j^1 s)(x))\\
&\quad = (0; \mathrm{d}(\varphi^j \circ j^1 \tilde{s})(x) - \mathrm{d}(\varphi^j \circ j^1 s)(x)).
\end{aligned}$$

Hence, for $4 \le i \le m$, we have

$$\begin{aligned}
\varphi^{i-3} \circ j^1 \tilde{s} &= \eta^1([X_3 + fY_3, X_i + fY_i]) = \eta^1([X_3, X_i] + (X_3 f)Y_i - (X_i f)Y_3)\\
&\quad + f\eta^1([Y_3, X_i] - [Y_i, X_3] + (Y_3 f)Y_i - (Y_i f)Y_3 + f[Y_3, Y_i]).
\end{aligned}$$

Taking into account that $f(x) = 0$ and $(\mathrm{d}f)_x = 0$ as $[f]^2_x \in \mathfrak{m}^2_x$, we obtain

$$\begin{aligned}
\mathrm{d}(\varphi^{i-3} \circ j^1 \tilde{s})(x) &= -(i_{[X_3, X_i]}\,\mathrm{d}\eta^1)_x + (L_{[X_3, X_i]}\eta^1)_x + (X_3 f)_x (L_{Y_i}\eta^1)_x\\
&\quad + \eta^1(Y_i)_x\,\mathrm{d}(X_3 f)(x) - (X_i f)_x (L_{Y_3}\eta^1)_x - \eta^1(Y_3)_x\,\mathrm{d}(X_i f)(x).
\end{aligned}$$

As $(\partial h_j{}^i/\partial x^k)(x) = (\partial f_j{}^i/\partial x^k)(x)\,\forall Z \in \mathfrak{X}(M)$ we have $\eta^1(Z)_x = \omega^1(Z)_x$, $(i_Z\,\mathrm{d}\eta^1)_x = (i_Z\,\mathrm{d}\omega^1)_x$ and $(L_Z\eta^1)_x = (L_Z\omega^1)_x$. Hence,

$$(\sigma_1)^{i-3}(j_x^1 s, [f]_x^2 \otimes v) = \omega^1(Y_i)\,\mathrm{d}(X_3 f)(x) - \omega^1(Y_3)\,\mathrm{d}(X_i f)(x). \tag{6}$$

We proceed similarly for the rest of components. □

3.2. Characteristics

A first-order PDE system is called a Cauchy–Kovalevskaya system if it can be written as follows:

$$\frac{\partial u^\beta}{\partial x^m} = \Phi^\beta\left(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^h}\right), \quad 1 \le h \le m-1. \tag{7}$$

If in (7) Φ^β are analytic on a neighbourhood of an analytic initial data then the Cauchy problem has a unique analytic solution (cf. [9, Theorem 1.1]).

Let $\pi : P \to M$ be a fibred manifold with $\dim M = m$, $\dim P = m+n$, let M' be the hypersurface on M described in a neighbourhood of $x_0 \in M'$ by the equation $x^m = 0$, where (x^i) is a coordinate system centred at x_0, and let

$$F^\beta(x^i, y^\alpha, y_i^\alpha) = 0 \tag{8}$$

be the PDE system that describes a fibred submanifold R^1 of $\pi^1 : J^1P \to M$ on a neighbourhood of $j_{x_0}^1 s$, $(x^i, y^\alpha, y_i^\alpha)$ being the induced coordinates system on J^1P. The point x_0 is said to be non-characteristic if and only if (8) can locally be written as a Cauchy–Kovalevskaya system; i.e. $\det(\partial F^\beta/\partial y_m^\alpha)(j_{x_0}^1 s) \neq 0$; otherwise, x_0 is said to be a characteristic point. The submanifold M' is said to be characteristic if every point $x_0 \in M'$ is characteristic.

In the PDE systems (1) and (2) the number of unknown functions is much greater than the number of equations; hence they are underdetermined systems. Accordingly, in order to study if the systems (1) and (2) are (generalized) Cauchy–Kovalevskaya systems we first need to introduce the notion of a characteristic submanifold for an arbitrary PDE system $R^1 \subset J^1P$ of codimension $\nu \le n$. Let M' be a hypersurface of M, $\dim M' = m-1$. A point $x_0 \in M'$ is said to be non-characteristic of R^1 if there exists $j_{x_0}^1 s \in R^1$ such that

$$\dim[(T_{x_0}^0 M' \otimes V_{s(x_0)}P) \cap T_{j_{x_0}^1 s} R^1] = n - \nu,$$

where $T_{x_0}^0 M' = \{w \in T_{x_0}^* M \,|\, w(v) = 0\,\forall v \in T_{x_0}M'\}$ and $T_{x_0}^0 M' \otimes V_{s(x_0)}P$ is identified with a subspace of $\varepsilon(T_{x_0}^* M \otimes V_{s(x_0)}P) \subset T_{j_{x_0}^1 s}(J^1P)$.

Proposition 3.4. *With the previous notations, let (x^i, y^α) be a fibred coordinate system for π such that in a neighbourhood of $x_0 \in M'$ the hypersurface M' is defined by $x^m = 0$ and let $F^k(x^i, y^\alpha, y_i^\alpha) = 0$, $1 \le k \le \nu$, be the equations describing R^1 in a neighbourhood of $j_{x_0}^1 s$. The point x_0 is non-characteristic if and only if*

$$\mathrm{rk}\left(\left(\frac{\partial F^k}{\partial y_m^\alpha}(j_{x_0}^1 s)\right)_\alpha^k\right) = \nu.$$

Proof. Taking into account that $(\partial/\partial y_m^\alpha)_{j_{x_0}^1 s}$ is a basis for $T_{x_0}^0 M' \otimes V_{s(x_0)} P$ and that $T_{j_{x_0}^1 s} R^1 = \{v \in T_{j_{x_0}^1 s}(J^1 P) | \mathrm{d}F^k(v) = 0, 1 \leq k \leq \nu\}$, we have

$$v \in (T_{x_0}^0 M' \otimes V_{s(x_0)} P) \cap T_{j_{x_0}^1 s} R^1 \Leftrightarrow \begin{cases} v = \lambda^\alpha \left(\dfrac{\partial}{\partial y_m^\alpha}\right)_{j_{x_0}^1 s}, \\ 0 = \dfrac{\partial F^k}{\partial y_m^\alpha}(j_{x_0}^1 s)\lambda^\alpha, \qquad 1 \leq k \leq \nu, \end{cases}$$

and hence $\dim[(T_{x_0}^0 M' \otimes V_{s(x_0)} P) \cap T_{j_{x_0}^1 s} R^1] = n - \mathrm{rk}(\partial F^k / \partial y_m^\alpha (j_{x_0}^1 s))$. □

Corollary 3.5. *If $x_0 \in M'$ is not a characteristic point, then there exist indices $j_1, \ldots, j_\nu$, with $1 \leq j_1 < \cdots < j_\nu \leq n$, such that the system R^1 can locally be written as $y_m^{j_k} = \Phi^k(x^i, y^\alpha, y_i^\alpha)$, $1 \leq k \leq \nu$, $(i, \alpha) \notin \{(m, j_1), \ldots, (m, j_\nu)\}$. If, for the sake of simplicity, we suppose $j_1 = 1, \ldots, j_\nu = \nu$, then the solutions to R^1 are given by*

$$\frac{\partial u^k}{\partial x^m} = \Phi^k\left(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^j}, \frac{\partial u^\gamma}{\partial x^m}\right), \quad 1 \leq k \leq \nu, \ \ 1 \leq j \leq m-1, \ \ \nu + 1 \leq \gamma \leq n,$$

and therefore R^1 is a generalized Cauchy–Kovalevskaya system where the functions u^γ, $\nu + 1 \leq \gamma \leq n$, remain indeterminate.

Proposition 3.6. *Every hypersurface of M is characteristic for the systems (1) and (2).*

Proof. We only prove the statement for the system (1), the other case being similar. Let (x^i) be a coordinate system such that M' is given by $x^m = 0$ in a neighbourhood of $x_0 \in M'$. We set $s(x_0) = (\lambda_1^i (\partial/\partial x^i)_{x_0}, \ldots, \lambda_m^i (\partial/\partial x^i)_{x_0})$, for scalars $\lambda_j^i \in \mathbb{R}$. According to Proposition 3.4, we have

$$\dim[(T_{x_0}^0 M' \otimes V_{s(x_0)}(FM)) \cap T_{j_{x_0}^1 s} R^1] = m^2 - r,$$

where r is the rank of the $3(m-2) \times m^2$-matrix $\Lambda = (\partial \varphi^k / \partial x_{m,b}^a (j_{x_0}^1 s))_{a,b}^k$, $\varphi = (\varphi^k)$ being the morphism in (3). First, assume $m \geq 4$ and let us prove that $r < 3(m - 2)$. Let C_{ab} be the columns of Λ, written in the opposite lexicographical order. From the equations of the submersion φ in Proposition 3.1, it follows that the rank of Λ coincides with the rank of the submatrix whose columns are $C_{a1}, \ldots, C_{am}$, that is,

$$\begin{pmatrix}
0 & 0 & -{\lambda_a}^1\lambda_4^m & {\lambda_a}^1\lambda_3^m & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -{\lambda_a}^1\lambda_m^m & 0 & \cdots & {\lambda_a}^1\lambda_3^m \\
0 & -{\lambda_a}^1\lambda_4^m & 0 & {\lambda_a}^1\lambda_2^m & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -{\lambda_a}^1\lambda_m^m & 0 & 0 & \cdots & {\lambda_a}^1\lambda_2^m \\
-2{\lambda_a}^1\lambda_3^m & 0 & 2{\lambda_a}^1\lambda_1^m + \sum_{l=4}^m {\lambda_a}^l\lambda_l^m & -{\lambda_a}^4\lambda_3^m & \cdots & -{\lambda_a}^m\lambda_3^m \\
-2{\lambda_a}^1\lambda_2^m & 2{\lambda_a}^1\lambda_1^m + \sum_{l=4}^m {\lambda_a}^l\lambda_l^m & 0 & -{\lambda_a}^4\lambda_2^m & \cdots & -{\lambda_a}^m\lambda_2^m \\
0 & -{\lambda_a}^1\lambda_3^m & {\lambda_a}^1\lambda_2^m & 0 & \cdots & 0 \\
0 & -{\lambda_a}^4\lambda_3^m & {\lambda_a}^4\lambda_2^m & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -{\lambda_a}^m\lambda_3^m & {\lambda_a}^m\lambda_2^m & 0 & \cdots & 0
\end{pmatrix}.$$

It is easily checked that the last $m-2$ rows are proportional to each other and the columns C_{a1} and C_{ab}, $4 \leq b \leq m$, are proportional to C_{11} and C_{1b}, respectively. Therefore, r is equal to the rank of the following matrix of order $(2m-3) \times (3m-2)$:

$$\begin{pmatrix}
0 & 0 & -{\lambda_a}^1\lambda_4^m & {\lambda_1}^1\lambda_3^m & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -{\lambda_a}^1\lambda_m^m & 0 & \cdots & {\lambda_1}^1\lambda_3^m \\
0 & -{\lambda_a}^1\lambda_4^m & 0 & {\lambda_1}^1\lambda_2^m & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -{\lambda_a}^1\lambda_m^m & 0 & 0 & \cdots & {\lambda_1}^1\lambda_2^m \\
-{\lambda_1}^1\lambda_3^m & 0 & 2{\lambda_a}^1\lambda_1^m + \sum_{l=4}^m {\lambda_a}^l\lambda_l^m & -{\lambda_1}^4\lambda_3^m & \cdots & -{\lambda_1}^m\lambda_3^m \\
-{\lambda_1}^1\lambda_2^m & 2{\lambda_a}^1\lambda_1^m + \sum_{l=4}^m {\lambda_a}^l\lambda_l^m & 0 & -{\lambda_1}^4\lambda_2^m & \cdots & -{\lambda_1}^m\lambda_2^m \\
0 & -\lambda_3^m & \lambda_2^m & 0 & \cdots & 0
\end{pmatrix}$$

and hence $r \leq 2m-3 < 3(m-2)$, as $m \geq 4$.

Next, suppose $m = 3$. Assuming ${\lambda_1}^1 \neq 0$, from the equations of the submersion φ in Proposition 3.1, we have

$$\operatorname{rk} \Lambda = \operatorname{rk} \begin{pmatrix} -{\lambda_a}^1\lambda_3^3 & 0 & {\lambda_a}^1\lambda_1^3 \\ -{\lambda_a}^1\lambda_2^3 & {\lambda_a}^1\lambda_1^3 & 0 \\ 0 & -{\lambda_a}^1\lambda_3^3 & {\lambda_a}^1\lambda_2^3 \end{pmatrix} = \operatorname{rk} \begin{pmatrix} \lambda_3^3 & 0 & \lambda_1^3 \\ \lambda_2^3 & \lambda_1^3 & 0 \\ 0 & -\lambda_3^3 & \lambda_2^3 \end{pmatrix} < 3,$$

thus concluding the proof. □

3.3. *Curvature of (1) and (2)*

Proposition 3.7. *Let φ be the morphism given in Proposition 3.1. Let $(U; x^i)$ be a coordinate system centred at a point $x \in U \subseteq M$, such that $x^i_j(s(x)) = \delta^i_j$. The system of covectors $w = (w^1, \dots, w^{3(m-2)})$, $w^k = \lambda^k_j(\mathrm{d}x^j)_x \in T^*_x$, belongs to $\mathrm{im}(\sigma_1)_{j^1_x s}$ if and only if the following equations hold*:

$$\lambda_2^{i-3} - \lambda_3^{m-6+i} + \lambda_i^{2m-3} = 0, \quad 4 \le i \le m,$$
$$2\lambda_1^{2m-3} + \sum_{h=4}^{m} \lambda_h^{2m-6+h} - \lambda_2^{2m-5} + \lambda_3^{2m-4} = 0. \tag{9}$$

Therefore, $\mathrm{rk}(\sigma_1)_{j^1_x s} = 3m^2 - 7m + 2 \,\forall j^1_x s \in J^1(FM)$.

Proof. From Eq. (6), and with the same notations as in Proposition 3.3, for $4 \le i \le m$ we have

$$\begin{aligned}(\sigma_1)^{i-3}(j^1_x s, [f]^2_x \otimes v) &= (\mathrm{d}x^1(Y_i)\,\mathrm{d}(X_3 f) - \mathrm{d}x^1(Y_3)\,\mathrm{d}(X_i f))(x) \\ &= \left(g^1_i\,\mathrm{d}\left(\delta^h_3 \frac{\partial f}{\partial x^h}\right) - g^1_3\left(\delta^h_i \frac{\partial f}{\partial x^h}\right)\right)(x) \\ &= \left(\left(g^1_i \frac{\partial^2 f}{\partial x^3 \partial x^j} - g^1_3 \frac{\partial^2 f}{\partial x^i \partial x^j}\right)(\mathrm{d}x^j)\right)(x).\end{aligned}$$

Let $t^i_{j,hk} = t^i_{j,I}$, $I = (h, k)$ be the coordinates of $[f]^2_x \otimes v \in S^2T^*_x \otimes E_x$ introduced in Section 2.2. From the above computation we have $(\sigma_1)^{i-3}_{j^1_x s} = (t^1_{i,3j} - t^1_{3,ij})(\mathrm{d}x^j)_x$, $4 \le i \le m$. Similarly, we obtain

$$\begin{aligned}(\sigma_1)^{m-6+i}_{j^1_x s} &= (t^1_{i,2j} - t^1_{2,ij})(\mathrm{d}x^j)_x, \quad 4 \le i \le m, \\ (\sigma_1)^{2m-5}_{j^1_x s} &= \left(2(t^1_{3,1j} - t^1_{1,3j}) + \sum_{h=4}^{m}(t^h_{3,hj} - t^h_{h,3j})\right)(\mathrm{d}x^j)_x, \\ (\sigma_1)^{2m-4}_{j^1_x s} &= \left(2(t^1_{2,1j} - t^1_{1,2j}) + \sum_{h=4}^{m}(t^h_{2,hj} - t^h_{h,2j})\right)(\mathrm{d}x^j)_x, \\ (\sigma_1)^{2m-3}_{j^1_x s} &= (t^1_{3,2j} - t^1_{2,3j})(\mathrm{d}x^j)_x, \\ (\sigma_1)^{2m-6+i}_{j^1_x s} &= (t^i_{3,2j} - t^i_{2,3j})(\mathrm{d}x^j)_x, \quad 4 \le i \le m.\end{aligned} \tag{10}$$

Accordingly, the system of covectors $\lambda^k_j(\mathrm{d}x^j)_x$, $1 \le k \le 3(m-2)$, belongs to $\mathrm{im}(\sigma_1)_{j^1_x s}$ if and only if the following system has a solution:

$$\lambda^{i-3}_j = t^1_{i,3j} - t^1_{3,ij}, \quad 4 \le i \le m, \qquad \lambda^{m-6+i}_j = t^1_{i,2j} - t^1_{2,ij}, \quad 4 \le i \le m,$$
$$\lambda^{2m-5}_j = 2(t^1_{3,1j} - t^1_{1,3j}) + \sum_{h=4}^{m}(t^h_{3,hj} - t^h_{h,3j}),$$

$$\lambda_j^{2m-4} = 2(t_{2,1j}^1 - t_{1,2j}^1) + \sum_{h=4}^{m} (t_{2,hj}^h - t_{h,2j}^h), \qquad \lambda_j^{2m-3} = t_{3,2j}^1 - t_{2,3j}^1,$$
$$\lambda_j^{2m-6+i} = t_{3,2j}^i - t_{2,3j}^i, \quad 4 \le i \le m. \tag{11}$$

Then, it is readily checked that Eq. (9) in the statement are necessary conditions for the system (11) to be compatible. Moreover, they are also sufficient as if (9) hold, then a particular solution to (11) is given by

$$t_{i,3j}^1 = \lambda_j^{i-3}, \qquad t_{3,ij}^i = \frac{1}{m-3}\lambda_j^{2m-5}, \quad j \ne 2,$$
$$t_{i,2j}^1 = \lambda_j^{m-6+i}, \qquad t_{2,ij}^i = \frac{1}{m-3}\lambda_j^{2m-4}, \quad j \ne 3,$$
$$t_{2,3i}^1 = -\lambda_3^{m-6+i}, \qquad t_{2,3i}^i = \frac{1}{m-3}\lambda_3^{2m-4}, \qquad t_{3,2i}^1 = -\lambda_2^{i-3},$$
$$t_{3,2i}^i = \lambda_i^{2m-6+i} + \frac{1}{m-3}\lambda_3^{2m-4}, \qquad t_{3,2j}^1 = \lambda_j^{2m-3}, \quad j = 1, 2 \text{ or } 3,$$
$$t_{3,2j}^i = \lambda_j^{2m-6+i}, \quad j \ne i, \qquad t_{h,jk}^1 = 0 \quad \text{otherwise},$$

where $4 \le i \le m$, and this completes the proof. □

Theorem 3.8. *The curvature of the system* (1), $\Omega : R^1 \to (T^* \otimes E')/\mathrm{im}\,\sigma_1$, $\Omega(j_x^1 s) = p_1(\varphi)(j_x^2 s) \bmod \mathrm{im}\,\sigma_1$, *vanishes.*

Proof. The curvature of (1) vanishes if and only if the system of covectors $p_1(\varphi)(j_x^2 s) = (\mathrm{d}(\varphi^1 \circ j^1 s)(x), \dots, \mathrm{d}(\varphi^{3(m-2)} \circ j^1 s)(x))$ belongs to $\mathrm{im}\,\sigma_1$, for every $j_x^1 s \in R^1$; that is, if and only if the system of covectors $\lambda_j^k (\mathrm{d}x^j)_x$, with $\lambda_j^k = (\partial(\varphi^k \circ j^1 s)/\partial x^j)(x)$, $1 \le k \le 3(m-2)$, satisfy the system (9).

Let (x^i) be a coordinate system such that $x_j^i(s(x)) = \delta_j^i$. We set

$$A_{b,c}^a = \left(\frac{\partial f_b^a}{\partial x^c}\right)(x), \qquad A_{b,ch}^a = \left(\frac{\partial^2 f_b^a}{\partial x^c \partial x^h}\right)(x). \tag{12}$$

As $j_x^1 s \in R^1$, from the system (1), we have

$$A_{i,a}^1 - A_{a,i}^1 = 0, \quad 4 \le i \le m, \quad a = 2, 3,$$
$$2(A_{a,1}^1 - A_{1,a}^1) + \sum_{h=4}^{m} (A_{a,h}^h - A_{h,a}^h) = 0, \quad a = 2, 3,$$
$$A_{3,2}^j - A_{2,3}^j = 0, \quad j \ne 2, 3 \tag{13}$$

and from Proposition 3.1, we obtain

$$\frac{\partial(\varphi^{i-3} \circ j^1 s)}{\partial x^j}(x) = -A_{k,j}^1 (A_{i,3}^k - A_{3,i}^k) + A_{3,j}^k A_{i,k}^1 - A_{i,j}^k A_{3,k}^1 + A_{i,3j}^1 - A_{3,ij}^1,$$
$$\frac{\partial(\varphi^{m-6+i} \circ j^1 s)}{\partial x^j}(x) = -A_{k,j}^1 (A_{i,2}^k - A_{2,i}^k) + A_{2,j}^k A_{i,k}^1 - A_{i,j}^k A_{2,k}^1 + A_{i,2j}^1 - A_{2,ij}^1,$$

$$\frac{\partial(\varphi^{2m-5}\circ j^1s)}{\partial x^j}(x) = 2[A^1_{k,j}(A^k_{1,3}-A^k_{3,1})+A^k_{1,j}A^1_{3,k}-A^k_{3,j}A^1_{1,k}+A^1_{3,1j}-A^1_{1,3j}] + \sum_{l=4}^{m}[A^l_{k,j}(A^k_{l,3}-A^k_{3,l})-A^k_{3,j}A^l_{l,k}+A^k_{l,j}A^l_{3,k}+A^l_{l,3j}-A^l_{3,lj}],$$

$$\frac{\partial(\varphi^{2m-4}\circ j^1s)}{\partial x^j}(x) = 2[A^1_{k,j}(A^k_{1,2}-A^k_{2,1})+A^k_{1,j}A^1_{2,k}-A^k_{2,j}A^1_{1,k}+A^1_{2,1j}-A^1_{1,2j}] + \sum_{l=4}^{m}[A^l_{k,j}(A^k_{l,2}-A^k_{2,l})-A^k_{2,j}A^l_{l,k}+A^k_{l,j}A^l_{2,k}-A^l_{l,2j}+A^l_{2,lj}],$$

$$\frac{\partial(\varphi^{2m-3}\circ j^1s)}{\partial x^j}(x) = -A^1_{k,j}(A^k_{3,2}-A^k_{2,3})+A^k_{2,j}A^1_{3,k}-A^k_{3,j}A^1_{2,k}+A^1_{3,2j}-A^1_{2,3j},$$

$$\frac{\partial(\varphi^{2m-6+i}\circ j^1s)}{\partial x^j}(x) = -A^i_{k,j}(A^k_{3,2}-A^k_{2,3})+A^k_{2,j}A^i_{3,k}-A^k_{3,j}A^i_{2,k}+A^i_{3,2j}-A^i_{2,3j}$$

with $4\le i\le m$. Taking into account Eq. (13), it is readily checked that $\lambda^k_j = \partial(\varphi^k\circ j^1s)/\partial x^j(x)$, $1\le k\le 3(m-2)$, satisfy the system (9). □

Theorem 3.9. *The curvature of the system* (2), $\Omega : \bar{R}^1 \to (T^*\otimes E')/\mathrm{im}\,\bar{\sigma}_1$, $\Omega(j^1_xs) = p_1(\bar{\varphi})(j^2_xs) \bmod \mathrm{im}\,\bar{\sigma}_1$, *vanishes*.

Proof. The result follows from Proposition 3.2 proceeding as in the proofs of Propositions 3.3 and 3.7 and Theorem 3.8. □

3.4. *Integrability of the systems (1) and (2)*

Proposition 3.10. *Let* φ *be the morphism given in* Proposition 3.1. *With the same notations as in* Section 2.3, *we set* $\mathfrak{g}_1 = \ker\sigma$, $\mathfrak{g}_2 = \ker\sigma_1$. *Then, we have* $\dim\mathfrak{g}_{1,u} = m^3-3(m-2)$ *and* $\dim\mathfrak{g}_{2,u} = (1/2)(m^4+m^3-6m^2+14m-4)$ *for every* $u\in FM$. *Similarly, let* $\bar{\varphi}$ *be the morphism given in* Proposition 3.2. *We set* $\bar{\mathfrak{g}}_1 = \ker\bar{\sigma}$, $\bar{\mathfrak{g}}_2 = \ker\bar{\sigma}_1$. *Then, for every* $u\in FM$ *we have* $\dim\bar{\mathfrak{g}}_{1,u} = m^3-3(m-1)$ *and* $\dim\bar{\mathfrak{g}}_{2,u} = (1/2)(m^4+m^3-6m^2+8m-2)$.

Proof. Given $u\in F_xM$, let (x^i, x^i_j) be such that $x^i_j(u) = \delta^i_j$. Hence with the same notations as in Section 2.2, by using the argument at the end of the proof of Proposition 3.1 we conclude that an element $\xi\in T^*_x\otimes E_x$, with coordinates $t^i_{j,k} = t^i_{j,k}(\xi)$, belongs to $\ker\sigma_u$, if and only if the following system of equations holds:

$$t^1_{i,a}-t^1_{a,i}=0,\quad 4\le i\le m,\quad a=2,3,$$
$$2(t^1_{a,1}-t^1_{1,a})+\sum_{h=4}^{m}(t^h_{a,h}-t^h_{h,a})=0,\quad a=2,3,$$
$$t^j_{3,2}-t^j_{2,3}=0,\quad j\ne 2,3. \tag{14}$$

Therefore, $\dim\mathfrak{g}_{1,u} = \dim\ker\sigma_u = m^3-3(m-2)$. From Proposition 3.7, we obtain

$\dim \mathfrak{g}_{2,u} = \dim \ker(\sigma_1)_{j^1_x s} = (1/2)(m^4 + m^3 - 6m^2 + 14m - 4)$. The proof of the second part of the statement follows similarly. □

Theorem 3.11. *For every $u \in F_x M$, there exists a quasi-regular basis of T_x for $\mathfrak{g}_1$ at u.*

Proof. For $\dim M = 3$, let us prove that (t_1, t_2, t_3), with $t_i = (\partial/\partial x^i)_x$, is a quasi-regular basis. In this basis the equations of the subspace $\mathfrak{g}_{1,u,(t_1)}$ of $T^*_x \otimes E_x$ are: $t^1_{3,1} = 0$, $t^1_{2,1} = 0$, $t^1_{2,3} = t^1_{3,2}$, and those of $\mathfrak{g}_{1,u,(t_1,t_2)}$ are: $t^1_{3,1} = 0$, $t^1_{3,2} = 0$ (see Eq. (14)). Hence,

$$\begin{aligned}\dim \mathfrak{g}_{1,u,(t_1)} + \dim \mathfrak{g}_{1,u,(t_1,t_2)} &= \dim(T^*_{x,(t_1)} \otimes E_x) - 3 + \dim(T^*_{x,(t_1,t_2)} \otimes E_x) - 2 \\ &= 2 \cdot 3^2 - 3 + 1 \cdot 3^2 - 2 = 22,\end{aligned}$$

and from Proposition 3.10, we also obtain $\dim \mathfrak{g}_{2,u} - \dim \mathfrak{g}_{1,u} = 22$.

For $\dim M = 4$, (t_1, t_2, t_3, t_4), $t_i = (\partial/\partial x^i)_x$, is a quasi-regular basis. In fact, in this basis the equations of $\mathfrak{g}_{1,u,(t_1)}$ are: $t^1_{3,4} = t^1_{4,3}$, $t^1_{2,4} = t^1_{4,2}$, $-2t^1_{3,1} = t^4_{3,4} - t^4_{4,3}$, $-2t^1_{2,1} = t^4_{2,4} - t^4_{4,2}$, $t^1_{2,3} = t^1_{3,2}$, $t^4_{2,3} = t^4_{3,2}$, those of $\mathfrak{g}_{1,u,(t_1,t_2)}$ are: $t^1_{3,4} = t^1_{4,3}$, $t^1_{4,2} = 0$, $-2t^1_{3,1} = t^4_{3,4} - t^4_{4,3}$, $t^4_{4,2} = 0$, $t^1_{3,2} = 0$, $t^4_{3,2} = 0$, and those of $\mathfrak{g}_{1,u,(t_1,t_2,t_3)}$ are: $t^1_{4,3} = 0$, $t^1_{4,2} = 0$, $t^4_{4,3} = 0$, $t^4_{4,2} = 0$ (see Eq. (14)). Hence, $\dim \mathfrak{g}_{1,u,(t_1)} + \dim \mathfrak{g}_{1,u,(t_1,t_2)} + \dim \mathfrak{g}_{1,u,(t_1,t_2,t_3)} = 80$, and from Proposition 3.10 we also obtain $\dim \mathfrak{g}_{2,u} - \dim \mathfrak{g}_{1,u} = 80$.

Finally, for $\dim M = m \geq 5$, let us consider the basis $(t_1, \ldots, t_m)$, given by $t_j = {a_j}^i (\partial/\partial x^i)_x$, $(a^i_j) \in \mathrm{GL}(m; \mathbb{R})$ with dual basis $\alpha^j = a^j_i (\mathrm{d}x^i)_x$. Let us denote by $\bar{t}^h_{i,k}$ the coordinates of $\xi \in T^*_x \otimes E_x$ with respect to the basis α^i in T^*_x and the usual basis in E_x; hence, $t^i_{kj} = {a_k}^h \bar{t}^i_{h,j}$. Let ${a_2}^m = {a_3}^m = 0$. The equations of $T^*_{x,(t_1,\ldots,t_{m-1})} \otimes E_x$ in $T^*_x \otimes E_x$ are $\bar{t}^i_{k,j} = 0$, $k \neq m$ and, therefore, $t^i_{k,j} = {a_k}^m \bar{t}^i_{m,j}$. Hence, the equations of $\mathfrak{g}_{1,u,(t_1,\ldots,t_{m-1})}$ are

$$ {a_i}^m \bar{t}^1_{m,b} = 0, \quad 4 \leq i \leq m, \quad b = 2, 3, \qquad 2{a_1}^m \bar{t}^1_{m,b} + \sum_{l=4}^{m} {a_l}^m \bar{t}^l_{m,b} = 0, \quad b = 2, 3. $$

By taking ${a_i}^m \neq 0$, $4 \leq i \leq m$, we have $\dim \mathfrak{g}_{1,u,(t_1,\ldots,t_{m-1})} = m^2 - 2(m-2)$. For $r \leq m-2$, the equations of $T^*_{x,(t_1,\ldots,t_r)} \otimes E_x$ in $T^*_x \otimes E_x$ are $\bar{t}^i_{k,j} = 0$, $1 \leq k \leq r$, and therefore $t^i_{k,j} = \sum_{h=r+1}^{m} {a_k}^h \bar{t}^i_{h,j}$. Hence, the equations of $\mathfrak{g}_{1,u,(t_1,\ldots,t_r)}$ are

$$\begin{aligned} &\sum_{h=r+1}^{m-1} ({a_b}^h \bar{t}^1_{h,i} - {a_i}^h \bar{t}^1_{h,b}) + {a_i}^m \bar{t}^1_{m,b} = 0, \quad 4 \leq i \leq m, \quad b = 2, 3, \\ &2\left(\sum_{h=r+1}^{m-1} ({a_1}^h \bar{t}^1_{h,b} - {a_b}^h \bar{t}^1_{h,1}) + {a_1}^m \bar{t}^1_{m,b}\right) \\ &+ \sum_{l=4}^{m} \left(\sum_{h=r+1}^{m-1} ({a_l}^h \bar{t}^l_{h,b} - {a_b}^h \bar{t}^l_{h,l}) + {a_l}^m \bar{t}^l_{m,b}\right) = 0, \quad b = 2, 3, \\ &\sum_{h=r+1}^{m-1} ({a_2}^h \bar{t}^j_{h,3} - {a_3}^h \bar{t}^j_{h,2}) = 0, \quad j \neq 2, 3. \end{aligned} \tag{15}$$

If ${a_2}^{m-1} \neq 0$, the $3(m-2)$ equations in (15) are independent as it is readily checked. Hence, $\dim \mathfrak{g}_{1,u,(t_1,\ldots,t_r)} = m^2(m-r) - 3(m-2)$ for $1 \leq r \leq m-2$. Therefore

$$\sum_{j=1}^{m-1} \dim \mathfrak{g}_{1,u,(t_1,\ldots,t_j)} = \sum_{j=1}^{m-2} \dim \mathfrak{g}_{1,u,(t_1,\ldots,t_j)} + \dim \mathfrak{g}_{1,u,(t_1,\ldots,t_{m-1})}$$
$$= \sum_{j=1}^{m-2} (m^2(m-j) - 3(m-2)) + m^2 - 2(m-2)$$
$$= \frac{1}{2}(m^4 - m^3) - 3m^2 + 10m - 8 = \dim \mathfrak{g}_{2,u} - \dim \mathfrak{g}_{1,u}. \qquad \square$$

Behaving as in the proof of Theorem 3.11 we obtain the following theorem.

Theorem 3.12. *For every $u \in F_xM$ there exists a quasi-regular basis of T_x for $\bar{\mathfrak{g}}_1$ at u.*

Theorem 3.13. *The PDE systems (1) and (2) are formally integrable.*

Proof. From Proposition 3.10 we deduce that $\mathfrak{g}_2$ is a vector bundle over R^1. Then, as the curvature of (1) vanishes by virtue of Theorem 3.8, we deduce that $\pi_1^2 : R^2 \to R^1$ is surjective and, in Theorem 3.11, we obtained a quasi-regular basis of T_x for $\mathfrak{g}_1$ at u. Therefore (see [2, IX, Theorems 2.14 and 2.16; 9, Theorem 1.3]) the system R^1 is formally integrable. Similarly, from Theorems 3.9 and 3.12 we obtain the corresponding result for $\bar{R}^1$. $\square$

Corollary 3.14. *Let R^1 (resp. $\bar{R}^1$) be the Euler–Lagrange equations defined by Ω^1_{23} (resp. Ω^1_{12}). For every $j^1_{x_0}s_0 \in R^1$ (resp. $j^1_{x_0}\bar{s}_0 \in \bar{R}^1$) there exists a section $s : U \to FM$ (resp. $\bar{s} : U \to FM$) of class C^∞ on a neighbourhood of x_0 such that: (i) s (resp. $\bar{s}$) is a solution to R^1 (resp. $\bar{R}^1$); i.e., $j^1_x s \in R^1$ (resp. $j^1_x \bar{s} \in \bar{R}^1$) $\forall x \in U$, and (ii) $j^1_{x_0}s = j^1_{x_0}s_0$ (resp. $j^1_{x_0}\bar{s} = j^1_{x_0}\bar{s}_0$).*

Proof. While M is not a manifold of class C^ω, once an open coordinate domain $(U; x^i)$ has been fixed, R^1 and $\bar{R}^1$ are analytic submanifolds in $J^1(FU)$ with respect to the C^ω structure induced by the coordinate system considered, as follows from the equations of the submersions φ and $\bar{\varphi}$ in Propositions 3.1 and 3.2, respectively. Hence, R^1 and $\bar{R}^1$ being formally integrable, we can apply Kuranishi's theorem (see [2, IX, Theorem 3.3]) in order to ensure the existence of a solution s (resp. $\bar{s}$) of class C^ω with respect to the coordinates chosen. $\square$

Corollary 3.14 ensures the existence of local solutions to Euler–Lagrange equations of Ω^1_{23} and Ω^1_{12}, but topological obstructions may exist to the existence of global solutions. For example, if $\dim M = 3$, then Eqs. (1) and (2) are equivalent to saying that $d\omega^1 = 0$ and $d\omega^1 = 0$, $d\omega^3 = 0$, respectively, where $(\omega^1, \omega^2, \omega^3)$ is the dual coframe and the existence of a closed, non-singular 1-form on a compact 3-manifold imposes restrictive conditions on its topology; for example, see [3,20].

3.5. Integrability of the Jacobi equations

Let F be an open fibred submanifold in a vector bundle $\pi : E \to M$ and let $\Gamma(M, F)$ be the Fréchet manifold of the sections of π (cf. [11, I, Sections 4 and 4.1.2]). The tangent space to $\Gamma(M, F)$ at a section s can be identified to $\Gamma(M, s^*VF)$, where $VF \subset TF$ denotes the vertical sub-bundle (cf. [11, I, Sections 4 and 4.3.3]). Moreover, there is a natural isomorphism $E \cong s^*VF$. Every $\xi \in \Gamma(M, E)$ is identified to the vector field X_ξ defined by $(X_\xi)_x f = (\mathrm{d} f(s(x) + t\xi(x))/\mathrm{d}t)|_{t=0}$.

We denote by $X^{(1)}$ the infinitesimal contact transformation associated to a vector field X in F; e.g., see [15,17].

Let $D : \Gamma(M, F) \to \Gamma(M, W)$ be a differential operator defined on the space of smooth sections of F taking values into another vector bundle $W \to M$. Then, the linearization of D along $s \in \Gamma(M, F)$ is the linear differential operator $D'_s : \Gamma(M, E) \to \Gamma(M, W)$ defined by (see [7, Section 3; 9, Section 1.7])

$$D'_s(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t} D(s + t\xi)\right|_{t=0} = \lim_{t\to 0} \frac{D(s + t\xi) - D(s)}{t}.$$

If Ω_m is a Lagrangian density on J^1F and Θ is its Poincaré–Cartan form (cf. [14]), then a section s of $\pi : F \to M$ is an extremal of Ω_m, if and only if it satisfies the so-called Hamilton–Cartan equations: $(j^1s)^*(i_Y \mathrm{d}\Theta) = 0 \, \forall Y \in \mathfrak{X}(J^1F)$ (see [15,17]). On a coordinate system $(x^i, y^\alpha, y^\alpha_i)$ the Hamilton–Cartan equations are equivalent to the following equations: $(j^1s)^*(i_{\partial/\partial y^\alpha} \mathrm{d}\Theta) = 0$.

Proposition 3.15. *With the same hypotheses and notations as above, we set* $W = \oplus^n \wedge^m T^*$ *and we denote by* $\Gamma(\pi U, F)$ *the space of smooth sections of* π *over an open subset* $\pi U \subseteq M$, $(U; x^i, y^\alpha)$ *being a fibred coordinate domain in F. Let* $H : \Gamma(\pi U, F) \to \Gamma(\pi U, W)$ *be the differential operator defined as follows*: $H(s) = (j^1s)^*(i_{\partial/\partial y^\alpha} \mathrm{d}\Theta)$. *The linearized operator* $H'_s : \Gamma(\pi U, E) \to \Gamma(\pi U, W)$ *at an extremal* s *of* Ω_m *is given by* $H'_s(\xi) = (j^1s)^*(i_{\partial/\partial y^\alpha} L_{X^{(1)}_\xi} \mathrm{d}\Theta)$.

Proof. We can assume ξ has compact support. Let Z be a vertical vector field on E with compact support such that $Z_{s(x)} = (X_\xi)_x \, \forall x \in \pi U$ (see [13, I, Theorem 5.7]). Let $S_t : \pi U \to E$ be $S_t(x) = s(x) + t\xi(x)$. There is a section $\psi_t : \pi U \to E$ such that $\varphi_t \circ s = S_t + t^2\psi_t$, φ_t being the flow of Z. Hence $(j^1S_t)^*(i_{\partial/\partial y^\alpha} \mathrm{d}\Theta) = (j^1s)^*(J^1\varphi_t)^*(i_{\partial/\partial y^\alpha} \mathrm{d}\Theta) + t^2\omega^\alpha_t$, for certain m-forms ω^α_t on M. Therefore $H'_s(\xi) = (j^1s)^*(L_{Z^{(1)}} i_{\partial/\partial y^\alpha} \mathrm{d}\Theta)$. As s is an extremal, $(j^1s)^*(i_{[Z^{(1)},\partial/\partial y^\alpha]} \mathrm{d}\Theta) = 0$, and we have $(j^1s)^*(L_{Z^{(1)}} i_{\partial/\partial y^\alpha} \mathrm{d}\Theta) = (j^1s)^* (i_{\partial/\partial y^\alpha} L_{X^{(1)}_\xi} \mathrm{d}\Theta)$. □

Let $\mathcal{S} \subset \Gamma(M, F)$ be the set of extremals of Ω_m. The Jacobi fields are the solutions to the linearized Hamilton–Cartan equation; that is, a Jacobi field along an extremal $s \in \mathcal{S}$ is a π-vertical vector field defined along s, $X \in \Gamma(M, s^*VF)$, satisfying the Jacobi equation $(j^1s)^*(i_Y L_{X^{(1)}} \mathrm{d}\Theta) = 0 \, \forall Y \in \mathfrak{X}(J^1E)$, which is proved to be locally equivalent to the equations $(j^1s)^*(i_{\partial/\partial y^\alpha} L_{X^{(1)}} \mathrm{d}\Theta) = 0$. Hence Jacobi fields along s can be thought of as being the tangent space at s to the "manifold" $\mathcal{S}$, and therefore we denote them by $T_s\mathcal{S}$.

Proposition 3.16. *With the same hypotheses as above, let $\Phi : J^1F \to W$ be a quasi-linear morphism, let $D : \Gamma(M, F) \to \Gamma(M, W)$ be the first-order operator $D = \Phi \circ j^1$ and let $D'_s = \Psi_s \circ j^1 : \Gamma(M, E) \to \Gamma(M, W)$ be the linearized operator of D along $s \in \Gamma(M, F)$, where $\Psi_s : J^1E \to W$ is the associated linear morphism. Then the symbol of Ψ_s over $F \subset E$ equals the symbol of Φ.*

Corollary 3.17. *With the same hypotheses and notations as in* Proposition 3.16, *we have* $\ker\sigma(\Psi_s) = \ker\sigma(\Phi) = \mathfrak{g}_1$ *and* $\ker\sigma_1(\Psi_s) = \ker\sigma_1(\Phi) = \mathfrak{g}_2$.

Set $W = \oplus^{m^2}(\wedge^m T^*)$. Let $H, \bar{H} : \Gamma(\pi U, FM) \to \Gamma(\pi U, W)$ be the differential operators defined by $H(s) = s^*(i_{\partial/\partial x^i_j}\, \mathrm{d}\Theta^1_{23})$, $\bar{H}(s) = s^*(i_{\partial/\partial x^i_j}\, \mathrm{d}\Theta^1_{12})$, where Θ^1_{23}, Θ^1_{12} are the Poincaré–Cartan forms associated to the densities Ω^1_{23}, Ω^1_{12} respectively (see Section 2.5). The forms Θ^1_{23}, Θ^1_{12} are π^1-projectable onto FM; for the proof of this result and the local expression of Θ^1_{23}, Θ^1_{12}, we refer the reader to [16, Theorem 3.1]. Hence, in the definition of H, $\bar{H}$, it is not necessary to consider the 1-jet prolongation of the section s. We claim that $H^{-1}\{0\} = \varphi^{-1}(M \times \{0\})$ (resp. $\bar{H}^{-1}\{0\} = \bar{\varphi}^{-1}(M \times \{0\})$), where φ (resp. $\bar{\varphi}$) is the morphism defined in Proposition 3.1 (resp. Proposition 3.2). In fact, as it is proved in [16, Section 3.3], the equations $\varphi^i(j^1_x s) = 0$, $1 \le i \le 3(m-2)$ (resp. $\bar{\varphi}^i(j^1_x s) = 0$, $1 \le i \le 3(m-1)$), describing $R^1 = \varphi^{-1}(M \times \{0\})$ (resp. $\bar{R}^1 = \bar{\varphi}^{-1}(M \times \{0\})$ in $J^1(FM)$), are equivalent to the Euler–Lagrange equations of Ω^1_{23} (resp. Ω^1_{12}).

Let $\mathcal{S}$ and $\bar{\mathcal{S}}$ be the sets of extremals of Ω^1_{23} and Ω^1_{12}, respectively, and let $E = \oplus^m T$. For every $s \in \mathcal{S}$, let $H'_s, \bar{H}'_s : \Gamma(M, E) \to \Gamma(M, W)$ be the linearized operators along s, i.e., $H'_s(\xi) = s^*(i_{\partial/\partial x^i_j} L_{X_\xi}\, \mathrm{d}\Theta^1_{23})$, $\bar{H}'_s(\xi) = s^*(i_{\partial/\partial x^i_j} L_{X_\xi}\, \mathrm{d}\Theta^1_{12})$, where X_ξ is the vector field associated with ξ, as defined at the beginning of this section. Jacobi fields can also be described by less equations than those expected. In fact, according to [16, formula (39)], we have $\ker H'_s = \ker(\psi_s \circ j^1)$, where $\psi_s : J^1E \to M \times \mathbb{R}^{3(m-2)}$ is given by

$$
\begin{aligned}
&\psi_s^{i-3}(j^1_x\xi) = (X^{(1)}_\xi \mathcal{L}^1_{3i} \circ j^1 s)(x), \quad 4 \le i \le m,\\
&\psi_s^{m-6+i}(j^1_x\xi) = (X^{(1)}_\xi \mathcal{L}^1_{2i} \circ j^1 s)(x),\\
&\psi_s^{2m-5}(j^1_x\xi) = \left(\left(2X^{(1)}_\xi \mathcal{L}^1_{31} + \sum_{l=4}^{m} X^{(1)}_\xi \mathcal{L}^l_{3l}\right) \circ j^1 s\right)(x), \quad 4 \le i \le m,\\
&\psi_s^{2m-4}(j^1_x\xi) = \left(\left(2X^{(1)}_\xi \mathcal{L}^1_{21} + \sum_{l=4}^{m} X^{(1)}_\xi \mathcal{L}^l_{2l}\right) \circ j^1 s\right)(x),\\
&\psi_s^{2m-3}(j^1_x\xi) = (X^{(1)}_\xi \mathcal{L}^1_{23} \circ j^1 s)(x),\\
&\psi_s^{2m-6+i}(j^1_x\xi) = (X^{(1)}_\xi \mathcal{L}^i_{23} \circ j^1 s)(x), \quad 4 \le i \le m.
\end{aligned}
\tag{16}
$$

For Ω^1_{12} we also have $\ker \bar{H}'_s = \ker(\bar{\psi}_s \circ j^1)$, where $\bar{\psi}_s : J^1E \to M \times \mathbb{R}^{3(m-1)}$ is given by

$$
\begin{aligned}
&\bar{\psi}_s^{i-2}(j^1_x\xi) = (X^{(1)}_\xi \mathcal{L}^1_{2i} \circ j^1 s)(x), \quad 3 \le i \le m,\\
&\bar{\psi}_s^{m-4+i}(j^1_x\xi) = (X^{(1)}_\xi \mathcal{L}^1_{1i} \circ j^1 s)(x), \quad 3 \le i \le m,
\end{aligned}
$$

$$\bar{\psi}_s^{2m-3}(j_x^1\xi) = \left(\left(\sum_{l=3}^{m} X_\xi^{(1)}\mathcal{L}_{2l}^l\right)\circ j^1 s\right)(x),$$

$$\bar{\psi}_s^{2m-2}(j_x^1\xi) = \left(\left(\sum_{l=3}^{m} X_\xi^{(1)}\mathcal{L}_{1l}^l\right)\circ j^1 s\right)(x),$$

$$\bar{\psi}_s^{2m-1}(j_x^1\xi) = (X_\xi^{(1)}\mathcal{L}_{12}^1 \circ j^1 s)(x),$$

$$\bar{\psi}_s^{2m-3+i}(j_x^1\xi) = (X_\xi^{(1)}\mathcal{L}_{12}^i \circ j^1 s)(x), \quad 3 \le i \le m. \tag{17}$$

In what follows, we denote by $R_s^1 = \ker \psi_s$ (resp. $\bar{R}_s^1 = \ker \bar{\psi}_s$) the PDEs defined by the Jacobi fields along an extremal s of Ω_{23}^1 (resp. Ω_{12}^1).

Proposition 3.18. *The fibred morphisms (16) and (17) are surjective. Hence, R_s^1 and $\bar{R}_s^1$ are vector sub-bundles.*

Proof. We prove that ψ_s is surjective; the proof for $\bar{\psi}_s$ is similar. Let $(U; x^i)$ be a coordinate system centred at a point $x \in U \subseteq M$, such that $x_j^i(s(x)) = \delta_j^i$. Let $X_\xi \in s^*VF$, $X_\xi = \sum_{i,j} \Phi_j^i E_j^{i*}|_s$, $\Phi_j^i \in C^\infty(M)$, where E_j^i denotes the standard basis of $\mathfrak{gl}(m;\mathbb{R})$; i.e., $(E_j^i)_k^h = \delta_i^h\delta_k^j$, and E_j^{i*} denotes the fundamental vector field (e.g., see [13, I, Section 4]) associated with $E_j^i \in \mathfrak{gl}(m;\mathbb{R})$. According to [16, formula (39)], expanding Eq. (16) we conclude that $j_x^1\xi \in R_s^1$ if and only if, denoting by $(X_1,\dots,X_m)$ the linear frame induced by the section $s: U \to FM$, for $4 \le i \le m$, the following equations hold:

$$(X_3\Phi_i^1 - X_i\Phi_3^1 + \Phi_3^h(\mathcal{L}_{hi}^1\circ j^1s) - \Phi_i^h(\mathcal{L}_{h3}^1\circ j^1s) - \Phi_h^1(\mathcal{L}_{3i}^h\circ j^1s))(x) = 0,$$

$$(X_2\Phi_i^1 - X_i\Phi_2^1 + \Phi_2^h(\mathcal{L}_{hi}^1\circ j^1s) - \Phi_i^h(\mathcal{L}_{h2}^1\circ j^1s) - \Phi_h^1(\mathcal{L}_{2i}^h\circ j^1s))(x) = 0,$$

$$2(X_1\Phi_3^1 - X_3\Phi_1^1 + \Phi_1^h(\mathcal{L}_{h3}^1\circ j^1s) - \Phi_3^h(\mathcal{L}_{h1}^1\circ j^1s) - \Phi_h^1(\mathcal{L}_{13}^h\circ j^1s))(x)$$
$$+\sum_{l=4}^{m}(X_l\Phi_3^l - X_3\Phi_l^l + \Phi_l^h(\mathcal{L}_{h3}^l\circ j^1s) - \Phi_3^h(\mathcal{L}_{hl}^l\circ j^1s) - \Phi_h^l(\mathcal{L}_{l3}^h\circ j^1s))(x) = 0,$$

$$2(X_1\Phi_2^1 - X_2\Phi_1^1 + \Phi_1^h(\mathcal{L}_{h2}^1\circ j^1s) - \Phi_2^h(\mathcal{L}_{h1}^1\circ j^1s) - \Phi_h^1(\mathcal{L}_{12}^h\circ j^1s))(x)$$
$$+\sum_{l=4}^{m}(X_l\Phi_2^l - X_2\Phi_l^l + \Phi_l^h(\mathcal{L}_{h2}^l\circ j^1s) - \Phi_2^h(\mathcal{L}_{hl}^l\circ j^1s) - \Phi_h^l(\mathcal{L}_{l2}^h\circ j^1s))(x) = 0,$$

$$(X_2\Phi_3^1 - X_3\Phi_2^1 + \Phi_2^h(\mathcal{L}_{h3}^1\circ j^1s) - \Phi_3^h(\mathcal{L}_{h2}^1\circ j^1s) - \Phi_h^1(\mathcal{L}_{23}^h\circ j^1s))(x) = 0,$$

$$(X_2\Phi_3^i - X_3\Phi_2^i + \Phi_2^h(\mathcal{L}_{h3}^i\circ j^1s) - \Phi_3^h(\mathcal{L}_{h2}^i\circ j^1s) - \Phi_h^i(\mathcal{L}_{23}^h\circ j^1s))(x) = 0. \tag{18}$$

Given $v = (\lambda^j) \in \mathbb{R}^{3(m-2)}$, $1 \le j \le 3(m-2)$, let $X_\xi = \sum_{i,j}\Phi_j^i E_j^{i*}|_s \in s^*VF$ be any vector field such that

$$\Phi_b^a(x) = 0, \qquad \frac{\partial\Phi_3^1}{\partial x^i}(x) = -\lambda^{i-3},$$

$$\frac{\partial\Phi_2^1}{\partial x^i}(x) = -\lambda^{m-6+i}, \qquad \frac{\partial\Phi_3^1}{\partial x^1}(x) = \frac{1}{2}\lambda^{2m-5},$$

$$\frac{\partial \Phi^1_2}{\partial x^1}(x) = \frac{1}{2}\lambda^{2m-4}, \qquad \frac{\partial \Phi^1_3}{\partial x^2}(x) = \lambda^{2m-3},$$
$$\frac{\partial \Phi^i_3}{\partial x^2}(x) = \lambda^{2m-6+i}, \qquad \frac{\partial \Phi^a_b}{\partial x^s}(x) = 0 \quad \text{otherwise}$$

for $4 \leq i \leq m$. Then, it is straightforward to see that $\psi_s(j^1_x\xi) = v$, and the result follows. □

Proposition 3.19. *The projections* $\pi^1_0 : R^1_s \to E$, $\pi^1_0(j^1_x\xi) = \xi_x$, $\bar{\pi}^1_0 : \bar{R}^1_s \to E$, $\bar{\pi}^1_0(j^1_x\xi) = \xi_x$, *are surjective.*

Proof. We prove that $\pi^1_0 : R^1_s \to E$ is surjective; the proof for $\bar{\pi}^1_0 : \bar{R}^1_s \to E$ is similar. Let $(U; x^i)$ be a coordinate system centred at a point $x \in U \subseteq M$, such that $x^i_j(s(x)) = \delta^i_j$. Let $X_\xi \in s^*VF$, $X_\xi = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$, $\Phi^i_j \in C^\infty(M)$. Set $B^i_{j,k} = (\partial \Phi^i_j / \partial x^k)(x)$, then the system (18) becomes

$$B^1_{i,3} - B^1_{3,i} = \mu^{i-3}, \quad 4 \leq i \leq m, \qquad B^1_{i,2} - B^1_{2,i} = \mu^{m-6+i}, \quad 4 \leq i \leq m,$$
$$2(B^1_{3,1} - B^1_{1,3}) + \sum_{l=4}^{m}(B^l_{3,l} - B^l_{l,3}) = \mu^{2m-5},$$
$$2(B^1_{2,1} - B^1_{1,2}) + \sum_{l=4}^{m}(B^l_{2,l} - B^l_{l,2}) = \mu^{2m-4},$$
$$B^1_{3,2} - B^1_{2,3} = \mu^{2m-3}, \qquad B^i_{3,2} - B^i_{2,3} = \mu^{2m-6+i}, \quad 4 \leq i \leq m, \tag{19}$$

where μ^j, $1 \leq j \leq 3(m-2)$, are linear functions on $\Phi^i_k(x)$. It is easy to check that, for any fixed values of $\Phi^i_k(x)$, the system (19) has a solution. □

Let $s = (X_1, \ldots, X_m) \in \mathcal{S}$, $X_j = f^i_j \partial/\partial x^i$ and let $(U; x^i)$ be a coordinate system centred at a point $x \in U \subseteq M$, such that $x^i_j(s(x)) = \delta^i_j$. Let $\xi \in E$ and let $X_\xi = \sum_{i,j} \Phi^i_j E^{i*}_j|_s \in s^*VF$. We set $B^i_{j,hk} = (\partial^2 \Phi^i_j / \partial x^h \partial x^k)(x)$. From (16) we deduce that the first prolongation of ψ_s, $p_1(\psi_s) : J^2E \to J^1(M \times \mathbb{R}^{3(m-2)})$, $p_1(\psi_s)(j^2_x\xi) = j^1_x(\psi_s \circ j^1\xi)$ is given by

$$(p_1(\psi_s))^{i-3}(j^2_x\xi) = B^1_{i,3k} - B^1_{3,ik} - \mu^{i-3}_k,$$
$$(p_1(\psi_s))^{m-6+i}(j^2_x\xi) = B^1_{i,2k} - B^1_{2,ik} - \mu^{m-6+i}_k,$$
$$(p_1(\psi_s))^{2m-5}(j^2_x\xi) = 2(B^1_{3,1k} - B^1_{1,3k}) + \sum_{l=4}^{m}(B^l_{3,lk} - B^l_{l,3k}) - \mu^{2m-5}_k,$$
$$(p_1(\psi_s))^{2m-4}(j^2_x\xi) = 2(B^1_{2,1k} - B^1_{1,2k}) + \sum_{l=4}^{m}(B^l_{2,lk} - B^l_{l,2k}) - \mu^{2m-4}_k,$$
$$(p_1(\psi_s))^{2m-3}(j^2_x\xi) = B^1_{3,2k} - B^1_{2,3k} - \mu^{2m-3}_k,$$
$$(p_1(\psi_s))^{2m-6+i}(j^2_x\xi) = B^i_{3,2k} - B^i_{2,3k} - \mu^{2m-6+i}_k \tag{20}$$

with $4 \leq i \leq m$, and where μ_k^j, $1 \leq j \leq 3(m-2)$, are linear functions on $j_x^1\xi$ whose coefficients are algebraic functions on $\partial f_b^a/\partial x^c(x)$ and $\partial^2 f_b^a/\partial x^c \partial x^d(x)$.

Lemma 3.20. *Set* $R_s^2 = \ker p_1(\psi_s)$. *With the same hypotheses and assumptions as above, for every* $j_x^2\xi \in R_s^2$, *the projection* $\pi_1^2(j_x^2\xi) = j_x^1\xi$ *satisfies the equations*:

$$\mu_2^{i-3} - \mu_3^{m-6+i} + \mu_i^{2m-3} = 0, \quad 4 \leq i \leq m,$$
$$2\mu_1^{2m-3} + \sum_{h=4}^{m} \mu_h^{2m-6+h} - \mu_2^{2m-5} + \mu_3^{2m-4} = 0. \tag{21}$$

Proof. From (20) we obtain that $j_x^2\xi \in J^2E$ belongs to R_s^2 if and only if the following system has a solution:

$$\mu_k^{i-3} = B^1_{i,3k} - B^1_{3,ik}, \quad 4 \leq i \leq m,$$
$$\mu_k^{m-6+i} = B^1_{i,2k} - B^1_{2,ik}, \quad 4 \leq i \leq m,$$
$$\mu_k^{2m-5} = 2(B^1_{3,1k} - B^1_{1,3k}) + \sum_{l=4}^{m}(B^l_{3,lk} - B^l_{l,3k}),$$
$$\mu_k^{2m-4} = 2(B^1_{2,1k} - B^1_{1,2k}) + \sum_{l=4}^{m}(B^l_{2,lk} - B^l_{l,2k}),$$
$$\mu_k^{2m-3} = B^1_{3,2k} - B^1_{2,3k}, \qquad \mu_k^{2m-6+i} = B^i_{3,2k} - B^i_{2,3k}, \quad 4 \leq i \leq m. \tag{22}$$

It is readily checked that Eq. (21) in the statement are necessary conditions for the system (22) to be compatible. □

Lemma 3.21. *Every element* $j_x^1\xi \in R_s^1$ *satisfies* (21).

Proof. We use the same notations as in formula (12), and we set $B^i_j = \Phi^i_j(x)$, $B^i_{j,k} = (\partial\Phi^i_j/\partial x^k)(x)$ as above. From Eqs. (16) and (18) we obtain the terms μ_2^{i-3}, μ_3^{m-6+i}, μ_i^{2m-3} in Eq. (20), which are given by

$$\mu_2^{i-3} = A^a_{3,2}B^1_{i,a} - A^a_{i,2}B^1_{3,a} + B^a_{3,2}\mathcal{L}^1_{ai}(j_x^1 s) - B^a_{i,2}\mathcal{L}^1_{a3}(j_x^1 s) - B^1_{a,2}\mathcal{L}^a_{3i}(j_x^1 s)$$
$$+ B^a_3\frac{\partial(\mathcal{L}^1_{ai}\circ j^1 s)}{\partial x^2}(x) - B^a_i\frac{\partial(\mathcal{L}^1_{a3}\circ j^1 s)}{\partial x^2}(x) - B^1_a\frac{\partial(\mathcal{L}^a_{3i}\circ j^1 s)}{\partial x^2}(x),$$
$$\mu_3^{m-6+i} = A^a_{2,3}B^1_{i,a} - A^a_{i,3}B^1_{2,a} + B^a_{2,3}\mathcal{L}^1_{ai}(j_x^1 s) - B^a_{i,3}\mathcal{L}^1_{a2}(j_x^1 s) - B^1_{a,3}\mathcal{L}^a_{2i}(j_x^1 s)$$
$$+ B^a_2\frac{\partial(\mathcal{L}^1_{ai}\circ j^1 s)}{\partial x^3}(x) - B^a_i\frac{\partial(\mathcal{L}^1_{a2}\circ j^1 s)}{\partial x^3}(x) - B^1_a\frac{\partial(\mathcal{L}^a_{2i}\circ j^1 s)}{\partial x^3}(x),$$
$$\mu_i^{2m-3} = A^a_{2,i}B^1_{3,a} - A^a_{3,i}B^1_{2,a} + B^a_{2,i}\mathcal{L}^1_{a3}(j_x^1 s) - B^a_{3,i}\mathcal{L}^1_{a2}(j_x^1 s) - B^1_{a,i}\mathcal{L}^a_{23}(j_x^1 s)$$
$$+ B^a_2\frac{\partial(\mathcal{L}^1_{a3}\circ j^1 s)}{\partial x^i}(x) - B^a_3\frac{\partial(\mathcal{L}^1_{a2}\circ j^1 s)}{\partial x^i}(x) - B^1_a\frac{\partial(\mathcal{L}^a_{23}\circ j^1 s)}{\partial x^i}(x)$$

with $4 \le i \le m$. By using $\mathcal{L}^a_{bc}(j^1_x s) = A^a_{c,b} - A^a_{b,c}$, we have

$$\begin{aligned}
&\mu_2^{i-3} - \mu_3^{m-6+i} + \mu_i^{2m-3} \\
&\quad = [\mathcal{L}^a_{23}(B^1_{i,a} - B^1_{a,i}) + \mathcal{L}^a_{2i}(B^1_{a,3} - B^1_{3,a}) + \mathcal{L}^a_{3i}(B^1_{2,a} - B^a_{a,2}) + \mathcal{L}^1_{ai}(B^a_{3,2} - B^a_{2,3}) \\
&\qquad + \mathcal{L}^1_{a3}(B^a_{2,i} - B^a_{i,2}) + \mathcal{L}^1_{a2}(B^a_{i,3} - B^a_{3,i})](j^1_x s) \\
&\qquad + \left[B^a_3 \left(\frac{\partial(\mathcal{L}^1_{ai} \circ j^1 s)}{\partial x^2} - \frac{\partial(\mathcal{L}^1_{a2} \circ j^1 s)}{\partial x^i} \right) + B^a_2 \left(\frac{\partial(\mathcal{L}^1_{a3} \circ j^1 s)}{\partial x^i} - \frac{\partial(\mathcal{L}^1_{ai} \circ j^1 s)}{\partial x^3} \right) \right. \\
&\qquad + B^a_i \left(\frac{\partial(\mathcal{L}^1_{a2} \circ j^1 s)}{\partial x^3} - \frac{\partial(\mathcal{L}^1_{a3} \circ j^1 s)}{\partial x^2} \right) \\
&\qquad \left. - B^1_a \left(\frac{\partial(\mathcal{L}^a_{3i} \circ j^1 s)}{\partial x^2} - \frac{\partial(\mathcal{L}^a_{2i} \circ j^1 s)}{\partial x^3} + \frac{\partial(\mathcal{L}^a_{23} \circ j^1 s)}{\partial x^i} \right) \right] (x). \qquad (23)
\end{aligned}$$

As s is an extremal, from Eq. (1) we have

$$\begin{aligned}
&[\mathcal{L}^a_{23}(B^1_{i,a} - B^1_{a,i}) + \mathcal{L}^a_{2i}(B^1_{a,3} - B^1_{3,a}) + \mathcal{L}^a_{3i}(B^1_{2,a} - B^1_{a,2}) + \mathcal{L}^1_{ai}(B^a_{3,2} - B^a_{2,3}) \\
&\quad + \mathcal{L}^1_{a3}(B^a_{2,i} - B^a_{i,2}) + \mathcal{L}^1_{a2}(B^a_{i,3} - B^a_{3,i})](j^1_x s) \\
&= [(\mathcal{L}^2_{23} - \mathcal{L}^1_{13})(B^1_{i,2} - B^1_{2,i}) + (\mathcal{L}^3_{23} + \mathcal{L}^1_{12})(B^1_{i,3} - B^1_{3,i}) \\
&\quad + (\mathcal{L}^2_{2i} + \mathcal{L}^3_{3i} - \mathcal{L}^1_{1i})(B^1_{2,3} - B^1_{3,2}) \\
&\quad + \mathcal{L}^j_{2i}(B^1_{j,3} - B^1_{3,j}) + \mathcal{L}^j_{3i}(B^1_{2,j} - B^1_{j,2}) + \mathcal{L}^1_{ji}(B^j_{3,2} - B^j_{2,3})](j^1_x s). \qquad (24)
\end{aligned}$$

As $j^1_x\xi \in R^1_s$, from (19) and again taking (1) into account, we obtain

$$\begin{aligned}
B^1_{i,a} - B^1_{a,i} &= -B^h_a \mathcal{L}^1_{hi}(j^1_x s) + B^h_i \mathcal{L}^1_{ha}(j^1_x s) + B^1_h \mathcal{L}^h_{ai}(j^1_x s) \\
&= \left[-B^1_a \mathcal{L}^1_{1i} - \sum_{h=4}^m B^h_a \mathcal{L}^1_{hi} + B^1_i \mathcal{L}^1_{1a} + B^1_2 \mathcal{L}^2_{ai} + B^1_3 \mathcal{L}^3_{ai} + \sum_{h=4}^m B^1_h \mathcal{L}^h_{ai} \right] (j^1_x s), \\
&\qquad a = 2, 3, \\
B^j_{3,2} - B^j_{2,3} &= -B^h_2 \mathcal{L}^j_{h3}(j^1_x s) + B^h_3 \mathcal{L}^j_{h2}(j^1_x s) + B^j_h \mathcal{L}^h_{23}(j^1_x s) \\
&= \left[-B^1_2 \mathcal{L}^j_{13} - \sum_{h=4}^m B^h_2 \mathcal{L}^j_{h3} + B^1_3 \mathcal{L}^j_{12} + \sum_{h=4}^m B^h_3 \mathcal{L}^j_{h2} + B^j_2 \mathcal{L}^2_{23} + B^j_3 \mathcal{L}^3_{23} \right] (j^1_x s), \\
&\qquad j \neq 2, 3
\end{aligned}$$

with $4 \le i \le m$. By substituting these expressions into (24) and factoring out B^a_b, Eq. (24) becomes

$$\begin{aligned}
&[(\mathcal{L}^2_{23} - \mathcal{L}^1_{13})(B^1_{i,2} - B^1_{2,i}) + (\mathcal{L}^3_{23} + \mathcal{L}^1_{12})(B^1_{i,3} - B^1_{3,i}) + \mathcal{L}^j_{2i}(B^1_{j,3} - B^1_{3,j}) \\
&\quad + (\mathcal{L}^2_{2i} + \mathcal{L}^3_{3i} - \mathcal{L}^1_{1i})(B^1_{2,3} - B^1_{3,2}) + \mathcal{L}^j_{3i}(B^1_{2,j} - B^1_{j,2}) + \mathcal{L}^1_{ji}(B^j_{3,2} - B^j_{2,3})](j^1_x s)
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg[B_2^1 \Bigg\{ \mathcal{L}_{3i}^3(\mathcal{L}_{13}^1 - \mathcal{L}_{23}^2) + \mathcal{L}_{3i}^2(\mathcal{L}_{23}^3 + \mathcal{L}_{12}^1) \\
&\quad + \sum_{j=4}^m (\mathcal{L}_{2i}^j \mathcal{L}_{3j}^2 + \mathcal{L}_{3i}^j(\mathcal{L}_{1j}^1 - \mathcal{L}_{2j}^2) - \mathcal{L}_{ji}^1 \mathcal{L}_{13}^j) \Bigg\} \\
&\quad + B_3^1 \Bigg\{ \mathcal{L}_{2i}^3(\mathcal{L}_{23}^2 - \mathcal{L}_{13}^1) - \mathcal{L}_{2i}^2(\mathcal{L}_{12}^1 + \mathcal{L}_{23}^3) \\
&\quad + \sum_{j=4}^m (\mathcal{L}_{2i}^j(\mathcal{L}_{3j}^3 - \mathcal{L}_{1j}^1) - \mathcal{L}_{3i}^j \mathcal{L}_{2j}^3 + \mathcal{L}_{ji}^1 \mathcal{L}_{12}^j) \Bigg\} + B_i^1\{\mathcal{L}_{12}^1(\mathcal{L}_{23}^2 - \mathcal{L}_{13}^1) \\
&\quad + \mathcal{L}_{13}^1(\mathcal{L}_{23}^3 + \mathcal{L}_{12}^1)\} + \sum_{h=4}^m \Bigg(B_2^h \Bigg\{ \mathcal{L}_{13}^1 \mathcal{L}_{hi}^1 + \sum_{j=4}^m (\mathcal{L}_{3i}^j \mathcal{L}_{hj}^1 - \mathcal{L}_{ji}^1 \mathcal{L}_{h3}^j) \Bigg\} \\
&\quad - B_3^h \Bigg\{ \mathcal{L}_{hi}^1 \mathcal{L}_{12}^1 + \sum_{j=4}^m (\mathcal{L}_{2i}^j \mathcal{L}_{hj}^1 - \mathcal{L}_{ji}^1 \mathcal{L}_{h2}^j) \Bigg\} \\
&\quad + B_h^1 \Bigg\{ \mathcal{L}_{2i}^h \mathcal{L}_{23}^2 + \mathcal{L}_{3i}^h \mathcal{L}_{23}^3) + \sum_{j=4}^m (\mathcal{L}_{2i}^j \mathcal{L}_{3j}^h - \mathcal{L}_{3i}^j \mathcal{L}_{2j}^h) \Bigg\} \Bigg) \Bigg] (j_x^1 s).
\end{aligned}
\tag{25}
$$

Again taking (1) into account, we have

$$
\begin{aligned}
&\Bigg[B_3^a \left(\frac{\partial(\mathcal{L}_{ai}^1 \circ j^1 s)}{\partial x^2} - \frac{\partial(\mathcal{L}_{a2}^1 \circ j^1 s)}{\partial x^i} \right) \\
&\quad + B_2^a \left(\frac{\partial(\mathcal{L}_{a3}^1 \circ j^1 s)}{\partial x^i} - \frac{\partial(\mathcal{L}_{ai}^1 \circ j^1 s)}{\partial x^3} \right) + B_i^a \left(\frac{\partial(\mathcal{L}_{a2}^1 \circ j^1 s)}{\partial x^3} - \frac{\partial(\mathcal{L}_{a3}^1 \circ j^1 s)}{\partial x^2} \right) \\
&\quad - B_a^1 \left(\frac{\partial(\mathcal{L}_{3i}^a \circ j^1 s)}{\partial x^2} - \frac{\partial(\mathcal{L}_{2i}^a \circ j^1 s)}{\partial x^3} + \frac{\partial(\mathcal{L}_{23}^a \circ j^1 s)}{\partial x^i} \right) \Bigg] (x) \\
&= [B_3^a\{\mathcal{L}_{2a}^b \mathcal{L}_{bi}^1 - \mathcal{L}_{ia}^1 \mathcal{L}_{12}^1 - \mathcal{L}_{2i}^b \mathcal{L}_{ba}^1\} + B_2^a\{\mathcal{L}_{ia}^1 \mathcal{L}_{13}^1 - \mathcal{L}_{3a}^b \mathcal{L}_{bi}^1 + \mathcal{L}_{3i}^b \mathcal{L}_{ba}^1\} \\
&\quad + B_i^a\{\mathcal{L}_{3a}^1 \mathcal{L}_{12}^1 - \mathcal{L}_{2a}^1 \mathcal{L}_{13}^1 - \mathcal{L}_{32}^b \mathcal{L}_{ba}^1\} - B_a^1\{\mathcal{L}_{3i}^b \mathcal{L}_{b2}^a - \mathcal{L}_{32}^b \mathcal{L}_{bi}^a - \mathcal{L}_{2i}^b \mathcal{L}_{b3}^a\}](j_x^1 s) \\
&= \Bigg[B_3^1 \Bigg\{ \sum_{h=4}^m (\mathcal{L}_{21}^h \mathcal{L}_{hi}^1 - \mathcal{L}_{3i}^h \mathcal{L}_{h2}^3 + \mathcal{L}_{2i}^h(\mathcal{L}_{h3}^3 - \mathcal{L}_{h1}^1) - \mathcal{L}_{2i}^2(\mathcal{L}_{21}^1 - \mathcal{L}_{23}^3)) \\
&\quad + \mathcal{L}_{2i}^3(\mathcal{L}_{32}^2 - \mathcal{L}_{31}^1) \Bigg\} + B_2^1 \Bigg\{ \sum_{h=4}^m (\mathcal{L}_{3i}^h(\mathcal{L}_{h1}^1 - \mathcal{L}_{h2}^2) - \mathcal{L}_{31}^h \mathcal{L}_{hi}^1 + \mathcal{L}_{2i}^h \mathcal{L}_{h3}^2) \\
&\quad + \mathcal{L}_{3i}^2(\mathcal{L}_{32}^3 - \mathcal{L}_{12}^1) - \mathcal{L}_{3i}^3(\mathcal{L}_{32}^2 - \mathcal{L}_{31}^1) \Bigg\} + B_i^1\{\mathcal{L}_{23}^2 \mathcal{L}_{21}^1 + \mathcal{L}_{23}^3 \mathcal{L}_{31}^1\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{h=4}^{m}\left(B_3^h\left\{\sum_{l=4}^{m}(\mathcal{L}_{2h}^l\mathcal{L}_{li}^1-\mathcal{L}_{2i}^l\mathcal{L}_{lh}^1)-\mathcal{L}_{ih}^1\mathcal{L}_{12}^1\right\}\right.\\
&\qquad -B_2^h\left\{\sum_{l=4}^{m}(\mathcal{L}_{3h}^l\mathcal{L}_{li}^1-\mathcal{L}_{3i}^l\mathcal{L}_{lh}^1)-\mathcal{L}_{ih}^1\mathcal{L}_{13}^1\right\}\\
&\qquad \left.\left.-B_h^1\left\{\sum_{l=4}^{m}(\mathcal{L}_{3i}^l\mathcal{L}_{l2}^h-\mathcal{L}_{2i}^l\mathcal{L}_{l3}^h)-\mathcal{L}_{32}^2\mathcal{L}_{2i}^h-\mathcal{L}_{32}^3\mathcal{L}_{3i}^h\right\}\right)\right](j_x^1 s).
\end{aligned}
$$

Hence, substituting (25) and the equations above into Eq. (23) we have $\mu_2^{i-3}-\mu_3^{m-6+i}+\mu_i^{2m-3}=0$. Similarly we obtain the last equation in (21). □

Proposition 3.22. *The projection $\pi_1^2 : R_s^2 \to R_s^1$ is surjective.*

Proof. We use the same notations as above. Given a point $j_x^1\xi \in R_s^1$, let μ_k^j, $1 \le j \le 3(m-2)$, be the linear functions corresponding to $j_x^1\xi$ introduced in (20). As Eq. (21) hold by virtue of Lemma 3.21, it is readily checked that a particular solution to (22) is given by

$$
\begin{aligned}
&B_{i,3j}^1=\mu_j^{i-3}, \qquad B_{3,ij}^i=\frac{1}{m-3}\mu_j^{2m-5}, \quad j\neq 2,\\
&B_{i,2j}^1=\mu_j^{m-6+i}, \qquad B_{2,ij}^i=\frac{1}{m-3}\mu_j^{2m-4}, \quad j\neq 3,\\
&B_{2,3i}^1=-\mu_3^{m-6+i}, \qquad B_{2,3i}^i=\frac{1}{m-3}\mu_3^{2m-4}, \qquad B_{3,2i}^1=-\mu_2^{i-3},\\
&B_{3,2i}^i=\mu_i^{2m-6+i}+\frac{1}{m-3}\mu_3^{2m-4}, \qquad B_{3,2j}^1=\mu_j^{2m-3}, \quad j=1,2\text{ or }3,\\
&B_{3,2j}^i=\mu_j^{2m-6+i}, \quad j\neq i, \qquad B_{h,jk}^l=0 \quad \text{otherwise}
\end{aligned}
$$

with $4\le i\le m$. □

Proposition 3.23. *$R_s^2 \subset J^2E$ is a vector sub-bundle.*

Proof. As $\pi_1^2 : R_s^2 \to R_s^1$ is surjective, in order to prove that R_s^2 is a vector sub-bundle, it suffices to prove that the rank of $\ker \pi_1^2$ is constant.

If $\pi_1^2(j_x^2\xi) = 0$, there exists $v\otimes\zeta \in S^2T_x^*\otimes E_x$ such that $\varepsilon(v\otimes\zeta) = j_x^2\xi$. Hence $j_x^2\xi \in R_s^2$ belongs to $\ker\pi_1^2$ if and only if $j_x^2\xi \in \ker\sigma_1(\psi_s)$. As $\ker\sigma_1(\psi_s) = \ker\sigma_1(\varphi)$ (see Proposition 3.16), from (10) we have that $j_x^2\xi \in \ker\sigma_1(\psi_s)$ if and only if it satisfies the following equations:

$$
\begin{aligned}
&E_k^{i-3}: \quad t_{i,3k}^1-t_{3,ik}^1=0, \quad 4\le i\le m,\\
&E_k^{m-6+i}: \quad t_{i,2k}^1-t_{2,ik}^1=0, \quad 4\le i\le m,\\
&E_k^{2m-5}: \quad 2(t_{3,1k}^1-t_{1,3k}^1)+\sum_{l=4}^{m}(t_{3,lk}^l-t_{l,3k}^l)=0,
\end{aligned}
$$

$$E_k^{2m-4}: \quad 2(t^1_{2,1k} - t^1_{1,2k}) + \sum_{l=4}^{m}(t^l_{2,lk} - t^l_{l,2k}) = 0, \qquad E_k^{2m-3}: \quad t^1_{3,2k} - t^1_{2,3k} = 0,$$
$$E_k^{2m-6+i}: \quad t^i_{3,2k} - t^i_{2,3k} = 0, \quad 4 \leq i \leq m, \tag{26}$$

where $t^i_{j,hk}$ are the coordinates of $\upsilon \otimes \zeta$ introduced in Section 2.2. The equations in (26) are not independent, as the following linear relations hold:

$$E_2^{i-3} - E_3^{m-6+i} + E_i^{2m-3} = 0, \quad 4 \leq i \leq m,$$
$$2E_1^{2m-3} + \sum_{h=4}^{m} E_h^{2m-6+h} - E_2^{2m-5} + E_3^{2m-4} = 0. \tag{27}$$

Thus $\operatorname{rk}\ker \pi^2_1 = \operatorname{rk}(S^2T^* \otimes E) - (3m(m-2) - (m-2))$. Hence

$$\operatorname{rk} R_s^2 = \operatorname{rk} R_s^1 + \operatorname{rk}\ker \pi^2_1 = m^2 \binom{m+1}{2} - m^3 - 2m^2 + 4m + 4. \qquad \square$$

From the first prolongation of $\bar{\psi}_s$, $p_1(\bar{\psi}_s) : J^2E \to J^1(M \times \mathbb{R}^{3(m-1)})$, proceeding similarly as in Lemmas 3.20 and 3.21 and Propositions 3.22 and 3.23, we obtain the following proposition.

Proposition 3.24. *$\bar{R}_s^2$ is a vector sub-bundle.*

Theorem 3.25. *The Jacobi equations along an arbitrary extremal $s : M \to FM$ of Ω^1_{23} (resp. $\bar{s} : M \to FM$ of Ω^1_{12}) are formally integrable. Hence, if M and s (resp. $\bar{s}$) are of class C^ω, then given a point $j^1_{x_0}\xi_0 \in R^1_s$ (resp. $j^1_{x_0}\bar{\xi}_0 \in \bar{R}^1_s$) there exists a Jacobi field X_ξ (resp. $X_{\bar{\xi}}$) defined along s (resp. $\bar{s}$) on a neighbourhood of x_0 such that $j^1_{x_0}\xi = j^1_{x_0}\xi_0$ (resp. $j^1_{x_0}\bar{\xi} = j^1_{x_0}\bar{\xi}_0$).*

Proof. The second part of the statement follows from the first part as a consequence of Kuranishi's theorem (see [2, IX, Theorem 3.3]) as if $s \in \mathcal{S}$ is C^ω, then the Jacobi equations are also C^ω.

Let us prove that the Jacobi equations along $s \in \mathcal{S}$ are formally integrable. According to Proposition 3.23, R_s^2 is a vector sub-bundle. From Proposition 3.22 we obtain that $\pi^2_1 : R^2_s \to R^1_s$ is surjective. And from Corollary 3.17 and Theorem 3.11 we have that $\forall \xi_x \in F_x \subset E_x$ there exists a quasi-regular basis of T_x for $\ker \sigma(\psi_s)$ at ξ_x. Hence, from [2, X, Theorem 1.6] the result follows. The proof for Ω^1_{12} is similar. $\square$

4. Pre-symplectic structure attached to Ω^1_{23} and Ω^1_{12}

In this section we assume that Ω^1_{23} and Ω^1_{12} have global extremals.

4.1. The small radical

Let Ω_m be a Lagrangian density on J^1P and let Θ be its Poincaré–Cartan form. Let $X, Y \in T_s\mathcal{S}$ be Jacobi vector fields defined along an extremal $s \in \mathcal{S}$ of Ω_m. Then, $\mathrm{d}[(j^1s)^*(i_{Y^{(1)}} i_{X^{(1)}}\,\mathrm{d}\Theta)] = 0$ (e.g., see [15]); that is, the $(m-1)$-form $i_{Y^{(1)}} i_{X^{(1)}}\,\mathrm{d}\Theta$ is closed along j^1s. The alternate bilinear map taking values in the space $Z^{m-1}(M)$ of closed $(m-1)$-forms on M,

$$(\omega_2)_s : T_s\mathcal{S} \times T_s\mathcal{S} \to Z^{m-1}(M), \qquad (\omega_2)_s(X, Y) = (j^1s)^*(i_{Y^{(1)}} i_{X^{(1)}}\,\mathrm{d}\Theta),$$

defines the pre-symplectic structure associated to Ω_m.

Let $s = (X_1, \ldots, X_m) : U \to FM$ be an extremal of Ω^1_{23}, $X_j = f^i_j \partial/\partial x^i$ with dual coframe $(\omega^1, \ldots, \omega^m)$, and let

$$X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s, \qquad Y = \sum_{i,j} \Upsilon^i_j E^{i*}_j|_s, \quad \Phi^i_j, \Upsilon^i_j \in C^\infty(M), \tag{28}$$

be two Jacobi fields where E^i_j denotes the standard basis of $\mathfrak{gl}(m;\mathbb{R})$. Then the pre-symplectic structure associated to Ω^1_{23} is given by (see [16, Proposition 3.13])

$$(\omega_2)_s(X, Y) = \begin{vmatrix} i_{X_2}\omega & i_{X_3}\omega & i_{X_h}\omega \\ \Upsilon^1_2 & \Upsilon^1_3 & \Upsilon^1_h \\ \Phi^h_2 & \Phi^h_3 & \Phi^h_h \end{vmatrix} + \begin{vmatrix} i_{X_2}\omega & i_{X_3}\omega & i_{X_h}\omega \\ \Upsilon^h_2 & \Upsilon^h_3 & \Upsilon^h_h \\ \Phi^1_2 & \Phi^1_3 & \Phi^1_h \end{vmatrix}, \tag{29}$$

where $\omega = \omega^1 \wedge \cdots \wedge \omega^m$. Similarly, the pre-symplectic structure associated to Ω^1_{12} is given by

$$(\bar{\omega}_2)_s(X, Y) = \begin{vmatrix} i_{X_1}\omega & i_{X_2}\omega & i_{X_h}\omega \\ \Upsilon^1_1 & \Upsilon^1_2 & \Upsilon^1_h \\ \Phi^h_1 & \Phi^h_2 & \Phi^h_h \end{vmatrix} + \begin{vmatrix} i_{X_1}\omega & i_{X_2}\omega & i_{X_h}\omega \\ \Upsilon^h_1 & \Upsilon^h_2 & \Upsilon^h_h \\ \Phi^1_1 & \Phi^1_2 & \Phi^1_h \end{vmatrix}. \tag{30}$$

Theorem 4.1. *Let $\mathcal{S}$ and $\bar{\mathcal{S}}$ be the sets of extremals of Ω^1_{23} and Ω^1_{12}, respectively, and let us consider an extremal $s \in \mathcal{S}$ (resp. $s \in \bar{\mathcal{S}}$). Assume M and s are of class C^ω. Then a Jacobi field $X \in T_s\mathcal{S}$ (resp. $X \in T_s\bar{\mathcal{S}}$) satisfies $i_X(\omega_2)_s = 0$ (resp. $i_X(\bar{\omega}_2)_S = 0$) if and only if it is an infinitesimal symmetry of Ω^1_{23} (resp. Ω^1_{12}).*

Proof. Let us first suppose that X is an infinitesimal symmetry of Ω^1_{23}; i.e., $L_{X^{(1)}}\Omega^1_{23} = 0$. Then, from [16, Theorem 3.7], we have $i_x\Theta^1_{23} = 0$. Hence for every $Y \in T_s\mathcal{S}$ we obtain, $0 = i_Y(L_X\Theta^1_{23}) = i_Y i_X\,\mathrm{d}\Theta^1_{23} + i_Y\,\mathrm{d}i_X\,\Theta^1_{23} = i_Y i_X\,\mathrm{d}\Theta^1_{23}$. Therefore, $i_X(\omega_2)_s(Y) = (\omega_2)_s(X, Y) = (j^1s)^*(i_X i_Y\,\mathrm{d}\Theta^1_{23}) = 0$.

Conversely, assume $(\omega_2)_s(X, Y) = 0$ for every $Y = \sum_{i,j} \Upsilon^i_j E^{i*}_j|_s \in T_s\mathcal{S}$. Let $(U; x^i)$ be a coordinate system centred at a point $x \in U \subseteq M$, such that $x^i_j(s(x)) = \delta^i_j$. Let $j^1_x\xi$ be a point in R^1_s with coordinates $\lambda^i_j, \lambda^i_{j,h}$. Since Jacobi field equations are formally integrable by virtue of Theorem 3.25, there exists a Jacobi field Y such that $\lambda^i_j = \Upsilon^i_j(x)$, $\lambda^i_{j,h} = (\partial\Upsilon^i_j/\partial x^h)(x)$.

According to Proposition 3.19, the projection $\pi_0^1 : R_s^1 \to E$, $\pi_0^1(j_x^1\xi) = \xi_x$ is surjective; hence the values λ_j^i can be taken arbitrarily. Let $X = \sum_{i,j} \Phi_j^i E_j^{i*}|_s$. From (29) we have

$$\begin{aligned}
0 &= \Phi_3^l(x)\lambda_2^1 - \Phi_2^l(x)\lambda_3^1 + \Phi_3^1(x)\lambda_2^l - \Phi_2^1(x)\lambda_3^l, \quad l \neq 2, 3,\\
0 &= \Phi_h^h(x)\lambda_3^1 - \Phi_3^h(x)\lambda_h^1 + \Phi_h^1(x)\lambda_3^h - \Phi_3^1(x)\lambda_h^h +\\
&\quad + \Phi_3^2(x)\lambda_2^1 - \Phi_2^2(x)\lambda_3^1 + \Phi_3^1(x)\lambda_2^2 - \Phi_2^1(x)\lambda_3^2, \quad l = 2,\\
0 &= -\Phi_h^h(x)\lambda_2^1 + \Phi_2^h(x)\lambda_h^1 - \Phi_h^1(x)\lambda_2^h + \Phi_2^1(x)\lambda_h^h + \Phi_3^3(x)\lambda_2^1\\
&\quad - \Phi_2^3(x)\lambda_3^1 + \Phi_3^1(x)\lambda_2^3 - \Phi_2^1(x)\lambda_3^3, \quad l = 3
\end{aligned}$$

and since λ_j^i are arbitrary, we obtain $\Phi_2^l(x) = \Phi_3^l(x) = 0$ for $l \neq 2, 3$, $\Phi_h^1(x) = 0$, for $h \neq 1$ and $2\Phi_1^1(x) + \sum_{l=4}^m \Phi_l^l(x) = 0$. Therefore, X is an infinitesimal symmetry (see [16, Proposition 3.3]).

For the density Ω_{12}^1, the proof is similar. □

4.2. *The large radical*

In this section we analyse the radical of the pre-symplectic structure modulo exact $(m-1)$-forms $B^{m-1}(M)$ for Ω_{23}^1 and Ω_{12}^1; that is,

$$(\varpi_2)_s : T_s\mathcal{S} \times T_s\mathcal{S} \to H^{m-1}(M;\mathbb{R}), \qquad (\varpi_2)_s(X,Y) = [(\omega_2)_s(X,Y)],$$
$$(\bar{\varpi}_2)_s : T_s\bar{\mathcal{S}} \times T_s\bar{\mathcal{S}} \to H^{m-1}(M;\mathbb{R}), \qquad (\bar{\varpi}_2)_s(X,Y) = [(\bar{\omega}_2)_s(X,Y)],$$

respectively, where $[\cdot]$ denotes the class modulo $B^{m-1}(M)$ of a closed form on M.

Lemma 4.2. *Let $s = (X_1, \dots, X_m) \in \mathcal{S}$ with dual basis $(\omega^1, \dots, \omega^m)$ and let X, Y be Jacobi fields as in (28) along s. If $\varphi^a = \Phi_b^a\omega^b$ and $\psi^a = \Upsilon_b^a\omega^b$, then the pre-symplectic structure attached to Ω_{23}^1 is given by*

$$\begin{aligned}
(\omega_2)_s(X,Y) &= 2\psi^1 \wedge \varphi^1 \wedge \omega^4 \wedge \cdots \wedge \omega^m\\
&\quad + \sum_{k=4}^m (-1)^k \omega^1 \wedge (\varphi^1 \wedge \psi^k - \psi^1 \wedge \varphi^k) \wedge \omega^4 \wedge \cdots \wedge \widehat{\omega^k} \wedge \cdots \wedge \omega^m,
\end{aligned}$$

where the symbol $(\hat{\ })$ means that the term is omitted.

Similarly, for $s \in \bar{\mathcal{S}}$ with dual basis $(\omega^1, \dots, \omega^m)$ the pre-symplectic structure attached to Ω_{12}^1 is given by

$$(\bar{\omega}_2)_s(X,Y) = \sum_{k=3}^m (-1)^k (\varphi^1 \wedge \psi^k - \psi^1 \wedge \varphi^k) \wedge \omega^3 \wedge \cdots \wedge \widehat{\omega^k} \wedge \cdots \wedge \omega^m.$$

Proof. According to (29) we have

$$\begin{aligned}
(\omega_2)_s(X,Y) &= 2[(\Upsilon_2^1\Phi_3^1 - \Upsilon_3^1\Phi_2^1)i_{X_1}\omega + (\Upsilon_3^1\Phi_1^1 - \Upsilon_1^1\Phi_3^1)i_{X_2}\omega\\
&\quad - (\Upsilon_2^1\Phi_1^1 - \Upsilon_1^1\Phi_2^1)i_{X_3}\omega] + \sum_{k=4}^m [(\Upsilon_3^1\Phi_k^k - \Upsilon_k^1\Phi_3^k)i_{X_2}\omega
\end{aligned}$$

$$-(\Upsilon_2^1\Phi_k^k - \Upsilon_k^1\Phi_2^k)i_{X_3}\omega + (\Upsilon_2^1\Phi_3^k - \Upsilon_3^1\Phi_2^k)i_{X_k}\omega$$
$$+ (\Upsilon_3^k\Phi_k^1 - \Upsilon_k^k\Phi_3^1)i_{X_2}\omega + (\Upsilon_2^k\Phi_k^1 - \Upsilon_k^k\Phi_2^1)i_{X_3}\omega$$
$$+ (\Upsilon_2^k\Phi_3^1 - \Upsilon_3^k\Phi_2^1)i_{X_k}\omega].$$

We set

$$W = \omega^4 \wedge \cdots \wedge \omega^m, \qquad W^i = \omega^4 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^m. \tag{31}$$

Then, we have

$$\begin{aligned}(\Upsilon_2^1\Phi_3^1 - \Upsilon_3^1\Phi_2^1)i_{X_1}\omega + (\Upsilon_3^1\Phi_1^1 - \Upsilon_1^1\Phi_3^1)i_{X_2}\omega - (\Upsilon_2^1\Phi_1^1 - \Upsilon_1^1\Phi_2^1)i_{X_3}\omega \\ = (\psi^1 \wedge \varphi^1)(X_2, X_3)\omega^2 \wedge \omega^3 \wedge W + (\psi^1 \wedge \varphi^1)(X_1, X_3)\omega^1 \wedge \omega^3 \wedge W \\ + (\psi^1 \wedge \varphi^1)(X_1, X_2)\omega^1 \wedge \omega^2 \wedge W = \psi^1 \wedge \varphi^1 \wedge W,\end{aligned}$$

and for $k \geq 4$,

$$\begin{aligned}(\Upsilon_3^1\Phi_k^k - \Upsilon_k^1\Phi_3^k)i_{X_2}\omega - (\Upsilon_2^1\Phi_k^k - \Upsilon_k^1\Phi_2^k)i_{X_3}\omega + (\Upsilon_2^1\Phi_3^k - \Upsilon_3^1\Phi_2^k)i_{X_k}\omega \\ = -(\psi^1 \wedge \varphi^k)(X_3, X_k)\omega^1 \wedge \omega^3 \wedge W - (\psi^1 \wedge \varphi^k)(X_2, X_k)\omega^1 \wedge \omega^2 \wedge W \\ +(-1)^{k-1}(\psi^1 \wedge \varphi^k)(X_2, X_3)\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge W^k \\ = -(\omega^1 \wedge \psi^1 \wedge \varphi^k)(X_1, X_3, X_k)\omega^3 \wedge (-1)^k\omega^k \wedge W^k \\ -(\omega^1 \wedge \psi^1 \wedge \varphi^k)(X_1, X_2, X_k)\omega^2 \wedge (-1)^k\omega^k \wedge W^k \\ +(-1)^{k-1}(\omega^1 \wedge \psi^1 \wedge \varphi^k)(X_1, X_2, X_3)\omega^2 \wedge \omega^3 \wedge W^k \\ = -(-1)^k\omega^1 \wedge \psi^1 \wedge \varphi^k \wedge W^k.\end{aligned}$$

Similarly,

$$\begin{aligned}(\Upsilon_3^k\Phi_k^1 - \Upsilon_k^k\Phi_3^1)i_{X_2}\omega - (\Upsilon_2^k\Phi_k^1 - \Upsilon_k^k\Phi_2^1)i_{X_3}\omega + (\Upsilon_2^k\Phi_3^1 - \Upsilon_3^k\Phi_2^1)i_{X_k}\omega \\ = -(-1)^k\omega^1 \wedge \psi^k \wedge \varphi^1 \wedge W^k,\end{aligned}$$

and substituting these expressions into the right-hand side of the first equation, we conclude.

For the density Ω_{12}^1, the proof is similar. □

4.2.1. dim $M = 3$

Theorem 4.3. *If* $\dim M = 3$ *and* $H^1(M; \mathbb{R}) = 0$, *then* $\varpi_2 = 0$ *and* $\bar{\varpi}_2 = 0$.

Proof. First we prove that for every Jacobi field X along an extremal $s \in \mathcal{S}$, there exists a vector field $Z \in \mathfrak{X}(M)$ such that

$$i_X(\omega_2)_s = i_{\tilde{Z}_s^v}(\omega_2)_s, \tag{32}$$

where $\tilde{Z}_s^v$ is the vertical component along s (see [16, 3.3.2]) of the natural lift $\tilde{Z}$ (see Section 2.1) of Z to the linear frame bundle. As a simple computation shows (again see [16, 3.3.2]) we have $\tilde{Z}_s^v = \sum_{r,l} \omega^r([X_l, Z])E_l^{r*}|_s$. As $\dim M = 3$, according to (29), the only components of a Jacobi field X involved in $i_X(\omega_2)_s$ are $\Phi_1^1, \Phi_2^1, \Phi_3^1$, where we use the same notations as in (28). Hence, we only need to construct a vector field Z such that $\Phi^1_l = \omega^1([X_l, Z])$ for $1 \le l \le 3$. In the present case, the Jacobi equations (18) (also see [16, Theorem 3.8]) can be written as

$$X_a\Phi^1_b - X_b\Phi^1_a - \Phi^1_h\omega^h([X_a, X_b]) = 0, \quad 1 \le a < b \le 3. \tag{33}$$

Hence $\mathrm{d}(\Phi^1_l\omega^l) = 0$. As $H^1(M; \mathbb{R}) = 0$, there exists a function ϕ^1 such that $\Phi^1_l\omega^l = \mathrm{d}\phi^1$. Set $Z = \phi^1 X_1$. Taking Eq. (1) into account we obtain

$$\omega^1([X_l, Z]) = \omega^1([X_l, \phi^1 X_1]) = \omega^1(X_l(\phi^1)X_1) + \phi^1\omega^1([X_l, X_1]) = X_l(\phi^1) = \Phi^1_l.$$

Similarly, given a Jacobi field $Y = \Upsilon^r_l E^{r*}_l|_s$, there exists $T \in \mathfrak{X}(M)$ such that $\Upsilon^1_l = \omega^1([X_l, T])$. Therefore, from Lemma 4.2 and taking account of the equations of the extremals (1), also see [16, Theorem 2.5], we obtain

$$\begin{aligned}(\omega_2)_s(X, Y) &= 2\psi^1 \wedge \varphi^1 = 2L_T\omega^1 \wedge L_Z\omega^1\\ &= 2(i_T\,\mathrm{d}\omega^1 + \mathrm{d}i_T\omega^1) \wedge (i_Z\,\mathrm{d}\omega^1 + \mathrm{d}i_Z\omega^1)\\ &= 2\,\mathrm{d}i_T\omega^1 \wedge \mathrm{d}i_Z\omega^1 = 2\,\mathrm{d}(i_T\omega^1 \wedge \mathrm{d}i_Z\omega^1).\end{aligned}$$

This concludes the proof for $(\omega_2)_s$.

For the pre-symplectic structure $(\bar{\omega}_2)_s$ the proof is similar. □

We remark that if M is compact, then the result in Theorem 4.3 simply follows by duality. If $H^1(M; \mathbb{R}) \neq 0$, then ϖ_2 and $\bar{\varpi}_2$ need not be zero. For example, if $M = (S^1)^3$ is the 3-torus with coordinates $\mathrm{d}x^i (\bmod 2\pi)$, then the dual basis $s = (X_1, X_2, X_3)$ of $(\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$ is an extremal for Ω^1_{23} and Ω^1_{12} (see [16, Proposition 2.7]), and using (33), we deduce that every vector field in (28) with constant coefficients, say $\Phi^i_j = \lambda^i_j \in \mathbb{R}$, $\Upsilon^i_j = \mu^i_j \in \mathbb{R}$ is a Jacobi field. We then obtain

$$(\varpi_2)_s(X, Y) = 2\sum_{a<b}(\lambda^1_b\mu^1_a - \lambda^1_a\mu^1_b)[\mathrm{d}x^a \wedge \mathrm{d}x^b],$$

and recalling that $[\mathrm{d}x^a \wedge \mathrm{d}x^b]$, $a < b$, is a basis for $H^2(M; \mathbb{R})$, the result follows.

4.2.2. dim $M \ge 4$

In this section we assume $M = N \times (a, b)$, where N is an orientable, connected and compact manifold of dimension $m - 1$, that is, a space-like hypersurface in M. Hence $H^{m-1}(M; \mathbb{R}) = H^{m-1}(N; \mathbb{R}) \cong \mathbb{R}$. Let $\iota : N \to M$, be the embedding $\iota(x) = (x, c)$, with $a < c < b$. Assume $s = (X_1, \dots, X_m)$ (with dual coframe $(\omega^1, \dots, \omega^m)$) is an extremal for Ω^1_{23} such that the vector fields $X_a = f^b_a\partial/\partial x^b$, $1 \le a \le m - 1$, are tangent to N and, furthermore, we have $X_m = \partial/\partial x^m$. Letting $\tilde{\omega}^i = \iota^*\omega^i$, we have $\tilde{\omega}^a = \sum_{h=1}^{m-1} \tilde{f}_h{}^a\,\mathrm{d}x^h$, $1 \le a \le m - 1$, where $\tilde{f}_h{}^a = f_h{}^a \circ \iota$, and $\tilde{\omega}^m = \iota^*(\mathrm{d}x^m) = 0$.

Lemma 4.4. *By using the notations introduced in (31), a linear frame s is an extremal for Ω^1_{23} if and only if its dual coframe satisfies the following equations*:

$$0 = \mathrm{d}\omega^1 \wedge \omega^1 \wedge W^i, \quad 4 \le i \le m, \tag{34}$$

$$0 = 2\,\mathrm{d}\omega^1 \wedge W + \omega^1 \wedge \mathrm{d}W. \tag{35}$$

Proof. The equations of the extremals for Ω^1_{23} (see [16, Theorem 2.3]) can be written as follows:

$$\mathrm{d}\omega^1 \wedge \omega^1 \wedge \omega^a \wedge W^i = 0, \quad 4 \leq i \leq m, \quad a = 2, 3, \tag{36a}$$

$$\omega^a \wedge [\mathrm{d}(\omega^1 \wedge W) + \mathrm{d}\omega^1 \wedge W] = 0, \quad a = 2, 3, \tag{36b}$$

$$\mathrm{d}\omega^j \wedge \omega^1 \wedge W = 0, \quad j \neq 2, 3. \tag{36c}$$

From (a) and (c) for $j = 1$, and taking the identity $\mathrm{d}\omega^1 \wedge \omega^1 \wedge \omega^1 \wedge W^i = 0$ into account, we obtain (34). By using (34), we have

$$\omega^1 \wedge [\mathrm{d}(\omega^1 \wedge W) + \mathrm{d}\omega^1 \wedge W] = 2(-1)^{i-1}\omega^i \wedge \omega^1 \wedge \mathrm{d}\omega^1 \wedge W^i = 0. \tag{37}$$

Moreover, taking (c) into account and $\omega^i \wedge W = 0$, for $4 \leq i \leq m$, we have

$$\omega^i \wedge [\mathrm{d}(\omega^1 \wedge W) + \mathrm{d}\omega^1 \wedge W] = \omega^i \wedge \omega^1 \wedge \mathrm{d}W = -\mathrm{d}\omega^i \wedge \omega^1 \wedge W = 0. \tag{38}$$

Hence, from (37), (36b) and (38) we obtain (35). □

Lemma 4.5. *Let $s = (X_1, \dots, X_m)$ be an extremal for Ω^1_{23} with dual coframe $(\omega^1, \dots, \omega^m)$. A vector field $X \in s^*VF$, $X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$, $\Phi^i_j \in C^\infty(M)$, is a Jacobi field for Ω^1_{23} along s if and only if the 1-forms $\varphi^a = \Phi^a_b \omega^b$ satisfy*

$$0 = \omega^1 \wedge \alpha^1 \wedge W^i, \quad 4 \leq i \leq m, \tag{39}$$

$$0 = 2\alpha^1 \wedge W + \sum_{k=4}^{m} (-1)^k \alpha^k \wedge \omega^1 \wedge W^k, \tag{40}$$

where

$$\alpha^h = \mathrm{d}\varphi^h - \varphi^1 \wedge i_{X_1}\,\mathrm{d}\omega^h - \sum_{l=4}^{m} \varphi^l \wedge i_{X_l}\,\mathrm{d}\omega^h. \tag{41}$$

Proof. From the Jacobi equations (see [16, Theorem 3.8]) and taking account of the following identities:

$$\begin{aligned} &X_j(\Phi^i_k) - X_k(\Phi^i_j) - \Phi^i_h \omega^h([X_j, X_k]) + \Phi^h_j \omega^i([X_h, X_k]) - \Phi^h_k \omega^i([X_h, X_j]) \\ &\quad = \mathrm{d}\varphi^i(X_j, X_k) - \varphi^h(X_j)\,\mathrm{d}\omega^i(X_h, X_k) + \varphi^h(X_k)\,\mathrm{d}\omega^i(X_h, X_j) \\ &\quad = (\mathrm{d}\varphi^i - \varphi^h \wedge i_{X_h}\,\mathrm{d}\omega^i)(X_j, X_k), \end{aligned}$$

we can rewrite down the Jacobi equations as follows:

$$0 = (\mathrm{d}\varphi^1 - \varphi^h \wedge i_{X_h}\,\mathrm{d}\omega^1)(X_a, X_i), \tag{42}$$

$$0 = 2(\mathrm{d}\varphi^1 - \varphi^h \wedge i_{X_h}\,\mathrm{d}\omega^1)(X_1, X_a) + \sum_{l=4}^{m} (\mathrm{d}\varphi^l - \varphi^h \wedge i_{X_h}\,\mathrm{d}\omega^l)(X_l, X_a), \tag{43}$$

$$0=(\mathrm{d}\varphi^j-\varphi^h\wedge i_{X_h}\,\mathrm{d}\omega^j)(X_2,X_3), \tag{44}$$

where $a=2,3$, $4\le i\le m$ and $j\ne 2,3$. Using the notation introduced in (41) and the equations for the extremals (see [16, Corollary 2.4]), Eq. (42) reads

$$\begin{aligned}0&=\alpha^1(X_a,X_i)-\sum_{b=2,3}(\varphi^b\wedge i_{X_b}\,\mathrm{d}\omega^1)(X_a,X_i)\\&=(\alpha^1\wedge\omega^1\wedge\omega^{a'}\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^m)(X_a,X_i,X_1,X_{a'},\dots,\hat{X}_i,\dots,X_m),\end{aligned}$$

where $a'=3$ if $a=2$ and $a'=2$ if $a=3$, and $4\le i\le m$. Hence, Eq. (42) is equivalent to

$$0=\alpha^1\wedge\omega^1\wedge\omega^a\wedge W^i,\quad a=2,3,\quad 4\le i\le m. \tag{45}$$

Proceeding similarly, Eqs. (43) and (44) can be written as follows:

$$0=2\alpha^1\wedge\omega^a\wedge W+\sum_{l=4}^m(-1)^l\alpha^l\wedge\omega^1\wedge\omega^a\wedge W^l,\quad a=2,3, \tag{46}$$

$$0=\omega^1\wedge\alpha^j\wedge W,\quad j\ne 2,3. \tag{47}$$

From (47), taking Eqs. (36c) and (39) into account, we obtain

$$\begin{aligned}0&=\mathrm{d}\varphi^1\wedge\omega^1\wedge(-1)^{i-1}\omega^i\wedge W^i+i_{X_1}\left(\varphi^1\wedge\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge W}_{=_{(36)}0}\right)\\&\quad-(i_{X_1}\varphi^1)\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge W}_{=_{(36)}0}+\varphi^1\wedge\mathrm{d}\omega^1\wedge i_{X_1}(\omega^1\wedge W)\\&\quad+\sum_{h=4}^m i_{X_h}\left(\varphi^h\wedge\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge W}_{=_{(36)}0}\right)-\sum_{h=4}^m(i_{X_h}\varphi^h)\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge W}_{=_{(36)}0}\\&\quad+\varphi^h\wedge\mathrm{d}\omega^1\wedge i_{X_h}(\omega^1\wedge(-1)^{i-1}\omega^i\wedge W^i)\\&=(-1)^i\omega^i\wedge\left\{(\mathrm{d}\varphi^1\wedge\omega^1+\varphi^1\wedge\mathrm{d}\omega^1)\wedge W^i-\sum_{h=4}^m\varphi^h\wedge\mathrm{d}\omega^1\wedge i_{X_h}(\omega^1\wedge W^i)\right\}\\&\quad-\sum_{h=4}^m(-1)^{h-1}\varphi^h\wedge\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge W^h}_{=_{(34)}0},\quad 4\le i\le m.\end{aligned} \tag{48}$$

Similarly, taking (34) and (35) into account, from (45) we have

$$0=\omega^a\wedge\left\{(\mathrm{d}\varphi^1\wedge\omega^1-\varphi^1\wedge\mathrm{d}\omega^1)\wedge W^i-\sum_{h=4}^m\varphi^h\wedge\mathrm{d}\omega^1\wedge i_{X_h}(\omega^1\wedge W^i)\right\}, \tag{49}$$

where $a = 2, 3$ and $4 \leq i \leq m$. And finally, taking (34) into account, we have

$$\omega^1 \wedge \left\{ (\mathrm{d}\varphi^1 \wedge \omega^1 - \varphi^1 \wedge \mathrm{d}\omega^1) \wedge W^i - \sum_{h=4}^{m} \varphi^h \wedge \mathrm{d}\omega^1 \wedge i_{X_h}(\omega^1 \wedge W^i) \right\}$$

$$= \varphi^1 \wedge \omega^1 \wedge \mathrm{d}\omega^1 \wedge W^i - \omega^1 \wedge \sum_{h=4}^{m} \varphi^h \wedge \mathrm{d}\omega^1 \wedge \omega^1 \wedge i_{X_h} W^i = 0.$$

Hence, from (48), (49) and equation above, we obtain

$$0 = (\mathrm{d}\varphi^1 \wedge \omega^1 - \varphi^1 \wedge \mathrm{d}\omega^1) \wedge W^i - \sum_{h=4}^{m} \varphi^h \wedge \mathrm{d}\omega^1 \wedge i_{X_h}(\omega^1 \wedge W^i), \quad 4 \leq i \leq m.$$

Proceeding similarly, from Eqs. (46) and (47) we obtain (40). □

We recall that in the present section we assume $M = N \times (a, b)$ and also that the linear frame $(X_1, \ldots, X_m)$ is adapted to such a splitting; see the beginning of Section 4.2.2. Because of this, if M admits an adapted holonomic linear frame, then N also admits a holonomic frame and hence it is necessarily diffeomorphic to the $(m-1)$-dimensional torus, as N is compact and connected. For a proof of this result, see [12, Lemma 2]. This justifies our assumption in the proposition below.

Proposition 4.6. *With the same hypothesis and notations as above, assume N is the $(m-1)$-dimensional torus, i.e., $N = (S^1)^{m-1}$, and let s be a holonomic section. A Jacobi field $X \in s^*VF$, $X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$, $\Phi^i_j \in C^\infty(M)$, belongs to the radical of the pre-symplectic structure $(\varpi_2)_s$, if and only if for $a = 1, m$, and for every $\alpha^1, \alpha^3, \alpha^4, \ldots, \alpha^{m-1} \in S^1$, the forms*

$$\tilde{\Phi}^a_3 \tilde{\omega}^2 \wedge \tilde{\omega}^3|_{\{\alpha^1\}\times S^1 \times S^1 \times \{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}}, \qquad \tilde{\Phi}^a_2 \tilde{\omega}^2|_{\{\alpha^1\}\times S^1\times\{\alpha^3\}\times\cdots\times\{\alpha^{m-1}\}} \tag{50}$$

are exact.

Proof. As s is a holonomic frame, we have $[X_i, X_j] = 0$. Parameterizing the points in N by $(\exp(\mathrm{i}x^1), \ldots, \exp(\mathrm{i}x^{m-1}))$, we can assume $\mathrm{d}x^j(X_k) = \delta^j_k$ for $j, k = 1, \ldots, m-1$. Let $\tilde{\Phi}^i_j(x) = \sum_{\alpha\in\mathbb{Z}^{m-1}} \Phi^i_{j,\alpha}(c)\chi_\alpha(x')$ be the Fourier series expansion of the functions $\tilde{\Phi}^i_j$, where $\chi_\alpha(x') = \exp[\mathrm{i}(\alpha_1 x^1 + \cdots + \alpha_{m-1}x^{m-1})]$, $x' = (x^1, \ldots, x^{m-1})$ and the coefficients $\Phi^i_{j,\alpha}(c)$ are given by

$$\Phi^i_{j,\alpha}(c) = \frac{1}{(2\pi)^{m-1}} \int_N \tilde{\Phi}^i_j(x)\chi_{-\alpha}(x')\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{m-1}.$$

As s is holonomic, by restricting (39) and (40) to N, the Jacobi equations become $\alpha_2\Phi^a_{3,\alpha}(c) = \alpha_3\Phi^a_{2,\alpha}(c)$, $a = 1, m$. Hence $\Phi^a_{3,\alpha}(c) = (\alpha_3/\alpha_2)\Phi^a_{2,\alpha}(c)$ if $\alpha_2 \neq 0$, and therefore

$$\tilde{\Phi}^a_3(x) = \sum_{\alpha_2\neq 0} \frac{\alpha_3}{\alpha_2}\Phi^a_{2,\alpha}(c)\chi_\alpha(x') + \phi^a_3(x^1, x^3, \ldots, x^{m-1}, c), \quad a = 1, m,$$

$$\phi^a_3(x^1, x^3, \ldots, x^{m-1}, c) = \sum_{\alpha\in\mathbb{Z}^{m-1}} \Phi^a_{3,(\alpha_1,0,\alpha_3,\ldots,\alpha_{m-1})}(c)\chi_{(\alpha_1,0,\alpha_3,\ldots,\alpha_{m-1})}(x'), \tag{51}$$

and if $\alpha_2 = 0$ and $\alpha_3 \neq 0$, then $\Phi^a_{2,\alpha}(c) = 0$, and therefore

$$\tilde{\Phi}^a_2(x) = \sum_{\alpha_2 \neq 0} \Phi^a_{2,\alpha}(c)\chi_\alpha(x') + \bar{\phi}^a_2(x^1, x^4, \dots, x^{m-1}, c), \quad a = 1, m,$$
$$\bar{\phi}^a_2(x^1, x^4, \dots, x^{m-1}, c) = \sum_{\alpha \in \mathbb{Z}^{m-1}} \Phi^a_{2,(\alpha_1,0,0,\alpha_4,\dots,\alpha_{m-1})}(c)\chi_{(\alpha_1,0,0,\alpha_4,\dots,\alpha_{m-1})}(x'). \tag{52}$$

Similarly, if $Y = \sum_{i,j} \Upsilon^i_j E^{i*}_j$ is a Jacobi field, we have

$$\tilde{\Upsilon}^a_3(x) = \sum_{\beta_2 \neq 0} \frac{\beta_3}{\beta_2} \Upsilon^a_{2,\beta}(c)\chi_\beta(x') + \psi^a_3(x^1, x^3, \dots, x^{m-1}, c), \quad a = 1, m, \tag{53}$$

$$\tilde{\Upsilon}^a_2(x) = \sum_{\beta_2 \neq 0} \Upsilon^a_{2,\beta}(c)\chi_\beta(x') + \bar{\psi}^a_2(x^1, x^4, \dots, x^{m-1}, c), \quad a = 1, m, \tag{54}$$

$$\psi^a_3(x^1, x^3, \dots, x^{m-1}, c) = \sum_{\beta \in \mathbb{Z}^{m-1}} \Upsilon^a_{3,(\beta_1,0,\beta_3,\dots,\beta_{m-1})}(c)\chi_{(\beta_1,0,\beta_3,\dots,\beta_{m-1})}(x'),$$
$$\bar{\psi}^a_2(x^1, x^4, \dots, x^{m-1}, c) = \sum_{\beta \in \mathbb{Z}^{m-1}} \Upsilon^a_{3,(\beta_1,0,0,\beta_4,\dots,\beta_{m-1})}(c)\chi_{(\beta_1,0,0,\beta_4,\dots,\beta_{m-1})}(x').$$

According to (29), the pre-symplectic structure associated to Ω^1_{23} is given by

$$\begin{aligned}(\tilde{\omega}_2)_s(X, Y) &= \iota^* \left(2\psi^1 \wedge \varphi^1 \wedge W + \sum_{k=4}^{m}(-1)^k \tilde{\omega}^1 \wedge (\varphi^1 \wedge \psi^k - \psi^1 \wedge \varphi^k) \wedge W^k\right) \\ &= (-1)^m \tilde{\omega}^1 \wedge (\tilde{\Phi}^1_b \tilde{\omega}^b \wedge \tilde{\Upsilon}^m_a \tilde{\omega}^a - \tilde{\Upsilon}^1_a \tilde{\omega}^a \wedge \tilde{\Phi}^m_b \tilde{\omega}^b) \wedge \iota^* W^m \\ &= (-1)^m (\tilde{\Phi}^1_2 \tilde{\Upsilon}^m_3 - \tilde{\Phi}^1_3 \tilde{\Upsilon}^m_2 - \tilde{\Upsilon}^1_2 \tilde{\Phi}^m_3 + \tilde{\Upsilon}^1_3 \tilde{\Phi}^m_2)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{m-1}.\end{aligned}$$

Expanding $\tilde{\Phi}^a_b$, $\tilde{\Upsilon}^a_b$, $a = 1, m$ and $b = 2, 3$ (see (51)–(54)), we obtain

$$\begin{aligned}\tilde{\Phi}^a_2 \tilde{\Upsilon}^{a'}_3 &= \sum_{\alpha_2 \cdot \beta_2 \neq 0} \frac{\beta_3}{\beta_2} \Phi^a_{2,\alpha}(c)\Upsilon^{a'}_{2,\beta}(c)\chi_{\alpha+\beta} + \sum_{\alpha_2 \neq 0} \Phi^a_{2,\alpha}(c)\psi^{a'}_3 \chi_\alpha \\ &\quad + \sum_{\beta_2 \neq 0} \frac{\beta_3}{\beta_2} \Upsilon^{a'}_{2,\beta}(c)\bar{\phi}^a_2 \chi_\beta + \bar{\phi}^a_2 \psi^{a'}_3,\end{aligned}$$

where $a' = m$ if $a = 1$, and $a' = 1$ if $a = m$, and similarly,

$$\begin{aligned}\tilde{\Upsilon}^a_2 \tilde{\Phi}^{a'}_3 &= \sum_{\alpha_2 \cdot \beta_2 \neq 0} \frac{\alpha_3}{\alpha_2} \Phi^{a'}_{2,\alpha}(c)\Upsilon^a_{2,\beta}(c)\chi_{\alpha+\beta} + \sum_{\beta_2 \neq 0} \Upsilon^a_{2,\beta}(c)\phi^{a'}_3 \chi_\beta \\ &\quad + \sum_{\alpha_2 \neq 0} \frac{\alpha_3}{\alpha_2} \Phi^{a'}_{2,\alpha}(c)\bar{\psi}^a_2 \chi_\alpha + \phi^{a'}_3 \bar{\psi}^a_2.\end{aligned}$$

By integrating $(\tilde{\omega}_2)_s(X,Y)$ on N, the terms

$$\sum_{a=1,m}\left(\sum_{\alpha_2\neq 0,\beta_2\neq 0}\frac{\beta_3}{\beta_2}\Phi^a_{2,\alpha}(c)\Upsilon^{a'}_{2,\beta}(c)\chi_{\alpha+\beta}-\sum_{\alpha_2\neq 0,\beta_2\neq 0}\frac{\alpha_3}{\alpha_2}\Phi^{a'}_{2,\alpha}(c)\Upsilon^{a}_{2,\beta}(c)\chi_{\alpha+\beta}\right)$$

vanish since, in this expression, either $\alpha+\beta\neq 0$ or $\alpha+\beta=0$ and then, the corresponding terms cancel out. Furthermore, from (51) and (53), we have

$$\sum_{a=1,m}\left(\sum_{\alpha_2\neq 0}\Phi^a_{2,\alpha}(c)\psi^{a'}_3\chi_\alpha-\sum_{\beta_2\neq 0}\Upsilon^a_{2,\beta}(c)\phi^{a'}_3\chi_\beta\right)$$
$$=\sum_{a=1,m}\left(\sum_{\alpha_2\neq 0}\sum_{\beta_2=0}\Phi^a_{2,\alpha}(c)\Upsilon^{a'}_{3,\beta}(c)\chi_{\alpha+\beta}-\sum_{\beta_2\neq 0}\sum_{\alpha_2=0}\Upsilon^a_{2,\beta}(c)\Phi^{a'}_{3,\alpha}(c)\chi_{\beta+\alpha}\right),$$

and, by again integrating on N, all the terms vanish, as $\alpha+\beta\neq 0$. Hence

$$\int_N(\tilde{\omega}_2)_s(X,Y)=\int_N(\bar{\phi}^1_2\psi^m_3-\bar{\psi}^m_2\phi^1_3-\bar{\psi}^1_2\phi^m_3+\bar{\phi}^m_2\psi^1_3)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{m-1}.$$

Letting $\psi^m_3=\bar{\psi}^m_2=\bar{\psi}^1_2=0$ and $\psi^1_3=\bar{\phi}^m_2$, we have

$$\int_N(\tilde{\omega}_2)_s(X,Y)=(2\pi)^2\int_{(S^1)^{m-3}}(\bar{\phi}^m_2)^2\,\mathrm{d}x^1\wedge\mathrm{d}x^4\wedge\cdots\wedge\mathrm{d}x^{m-1},$$

and therefore $\bar{\phi}^m_2=0$. Similarly, by taking $\psi^1_3=\bar{\psi}^m_2=\bar{\psi}^1_2=0$ and $\psi^m_3=\bar{\phi}^1_2$, we obtain $\bar{\phi}^1_2=0$. For $a=1,m$, let

$$\phi^a_3=\sum_{h\neq 0}\phi^a_{3,h}(x^1,x^4,\ldots,x^{m-1},c)\exp(\mathrm{i}hx^3)+\phi^a_{3,0}(x^1,x^4,\ldots,x^{m-1},c)$$

be the Fourier series expansion of the functions ϕ^a_3 with respect to x^3. We have

$$\int_N(\tilde{\omega}_2)_s(X,Y)=-2\pi\int_{(S^1)^{m-2}}(\bar{\psi}^m_2\phi^1_3+\bar{\psi}^1_2\phi^m_3)\,\mathrm{d}x^1\wedge\mathrm{d}x^3\wedge\cdots\wedge\mathrm{d}x^{m-1}$$
$$=-(2\pi)^2\int_{(S^1)^{m-3}}(\bar{\psi}^m_2\phi^1_{3,0}+\bar{\psi}^1_2\phi^m_{3,0})\,\mathrm{d}x^1\wedge\mathrm{d}x^4\wedge\cdots\wedge\mathrm{d}x^{m-1}.$$

Taking $\bar{\psi}^m_2=0$, $\bar{\psi}^1_2=\phi^m_{3,0}$, we obtain $\phi^m_{3,0}=0$, and taking $\bar{\psi}^1_2=0$, $\bar{\psi}^m_2=\phi^1_{3,0}$, we obtain $\phi^1_{3,0}=0$. Hence, the Jacobi field $X=\sum_{i,j}\Phi^i_jE^{i*}_j|_s$ belongs to the radical of $(\varpi_2)_s$ if and only if the functions $\tilde{\Phi}^a_3$, $\tilde{\Phi}^a_2$, $a=1,m$, satisfy

$$\tilde{\Phi}^a_3(x)=\sum_{\alpha_2\neq 0}\frac{\alpha_3}{\alpha_2}\Phi^a_{2,\alpha}(c)\chi_\alpha(x')+\sum_{h\neq 0}\phi^a_{3,h}(x^1,x^4,\ldots,x^{m-1},c)\exp(\mathrm{i}hx^3),$$
$$\tilde{\Phi}^a_2(x)=\sum_{\alpha_2\neq 0}\Phi^a_{2,\alpha}(c)\chi_\alpha(x').$$

From these expressions we conclude that the integrals over $S^1 \times S^1$ and over S^1, respectively, of the forms in (50), vanish. Hence such forms are exact.

Conversely, assume that the forms in (50) are exact. Therefore, from Eq. (51) we have

$$\begin{aligned}
0 &= \int_{\{\alpha^1\}\times S^1\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \tilde{\Phi}_3^a \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \sum_{\alpha_2\neq 0} \frac{\alpha_3}{\alpha_2} \int_{\{\alpha^1\}\times S^1\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \Phi_{2,\alpha}^a(c)\chi_\alpha(x')\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&\quad + \int_{\{\alpha^1\}\times S^1\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \phi_3^a(x^1, x^3, \dots, x^{m-1}, c)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= 2\pi \int_{\{\alpha^1\}\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \phi_3^a(x^1, x^3, \dots, x^{m-1}, c)\, \mathrm{d}x^3,
\end{aligned}$$

and recalling the Fourier series expansion for $\phi_3^a(x^1, x^3, \dots, x^{m-1}, c)$ with respect to x^3, we have

$$\begin{aligned}
0 &= 2\pi \int_{\{\alpha^1\}\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \phi_3^a(x^1, x^3, \dots, x^{m-1}, c)\, \mathrm{d}x^3 \\
&= 2\pi \int_{\{\alpha^1\}\times S^1\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \sum_{h\neq 0} \phi_{3,h}^a(x^1, x^4, \dots, x^{m-1}, c)\exp(\mathrm{i}hx^3)\, \mathrm{d}x^3 \\
&\quad + 2\pi \int_{\{\alpha^1\}\times S^1\times S^1\times\{\alpha^4\}\times\cdots\times\{\alpha^{m-1}\}} \phi_{3,0}^a(x^1, x^4, \dots, x^{m-1}, c)\, \mathrm{d}x^3 \\
&= (2\pi)^2 \phi_{3,0}^a(\alpha^1, \alpha^4, \dots, \alpha^{m-1}, c).
\end{aligned}$$

Similarly, from (52) we have

$$\begin{aligned}
0 &= \int_{\{\alpha^1\}\times S^1\times\{\alpha^3\}\times\cdots\times\{\alpha^{m-1}\}} \tilde{\Phi}_2^a \, \mathrm{d}x^2 \\
&= \int_{\{\alpha^1\}\times S^1\times\{\alpha^3\}\times\cdots\times\{\alpha^{m-1}\}} \sum_{\alpha_2\neq 0} \Phi_{2,\alpha}^a(c)\chi_\alpha(x')\, \mathrm{d}x^2 \\
&\quad + \int_{\{\alpha^1\}\times S^1\times\{\alpha^3\}\times\cdots\times\{\alpha^{m-1}\}} \bar{\phi}_2^a(x^1, x^4, \dots, x^{m-1}, c)\, \mathrm{d}x^2 \\
&= (2\pi)^2 \bar{\phi}_2^a(\alpha^1, \alpha^4, \dots, \alpha^{m-1}, c),
\end{aligned}$$

and this finishes the proof. □

Similarly, for the density Ω_{12}^1 we obtain the following lemma.

Lemma 4.7. *We set $\bar{W} = \omega^3 \wedge \cdots \wedge \omega^m$ and $\bar{W}^i = \omega^3 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^m$. A linear frame s is an extremal for Ω_{12}^1 if and only if its dual coframe satisfies the following equations*:

$$\mathrm{d}\bar{W} = 0, \qquad \mathrm{d}\omega^1 \wedge \bar{W}^i = 0, \quad 3 \leq i \leq m.$$

Lemma 4.8. *Let $s = (X_1, \dots, X_m)$ be an extremal for Ω^1_{12} with dual coframe $(\omega^1, \dots, \omega^m)$. A vector field $X \in s^*VF$, $X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$, $\Phi^i_j \in C^\infty(M)$, is a Jacobi field for Ω^1_{12} along s if and only if the 1-forms $\varphi^a = \Phi^a_b \omega^b$ satisfy*

$$0 = \mathrm{d}\varphi^1 \wedge \bar{W}^i - \sum_{l=1}^{m} \varphi^l \wedge i_{X_l} \mathrm{d}\omega^1 \wedge \bar{W}^i, \quad 3 \le i \le m,$$

$$0 = \sum_{k=3}^{m} (-1)^k \left(\mathrm{d}\varphi^k - \sum_{l=1}^{m} \varphi^l \wedge i_{X_l} \mathrm{d}\omega^k \right) \wedge \bar{W}^k.$$

Proposition 4.9. *With the same hypothesis and notations as above, assume N is the $(m-1)$-dimensional torus, i.e., $N = (S^1)^{m-1}$, and let s be a holonomic section. A Jacobi field $X \in s^*VF$, $X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$, $\Phi^i_j \in C^\infty(M)$, belongs to the radical of the pre-symplectic structure $(\bar{\varpi}_2)_s$, if and only if for $a = 1, m$, and for every $\alpha^2, \alpha^3, \alpha^4, \dots, \alpha^{m-1} \in S^1$, the following forms are exact*:

$$\tilde{\Phi}^a_2 \tilde{\omega}^1 \wedge \tilde{\omega}^2|_{S^1 \times S^1 \times \{\alpha^3\} \times \cdots \times \{\alpha^{m-1}\}}, \qquad \tilde{\Phi}^a_1 \tilde{\omega}^1|_{S^1 \times \{\alpha^2\} \times \cdots \times \{\alpha^{m-1}\}}.$$

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